The ℓ-adic trace formula for dg-categories and Bloch's conductor conjecture

Bertrand Toën*and Gabriele Vezzosi

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Abstract

Building on [BRTV], we present an ℓ-adic trace formula for smooth and proper dg-categories over some base $\mathbb{E}_\infty$-algebra B. We also give a variant when B is only an $\mathbb{E}_2$-algebra. As an application of this trace formula, we also propose a strategy of proof of Bloch's conductor conjecture. This is a research announcement and detailed proofs will appear elsewhere.

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Introduction

This is a research announcement of results whose details and proofs will appear in forthcoming papers.

The purpose of this note is to prove a trace formula in étale cohomology in the non-commutative setting. Here non-commutative spaces will be modeled by dg-categories and our trace formula reads as an equality between on the one side, a certain virtual number of fixed points of an endomorphism, and on the other side the trace of the induced endomorphism on the ℓ-adic cohomology spaces recently introduced in [BRTV] (following the main idea of topological realizations of dg-categories of [B]). As an application of our trace formula we sketch a proof of the Bloch's conductor formula [Bl] under the hypothesis that the monodromy is unipotent. We also present a general strategy in order to prove the general case.

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In order to describe our construction, we start by recalling some constructions and results from [BRTV]. In particular, for A an excellent ring, B an $\mathbb{E}_\infty$ A-algebra, and ℓ a prime invertible on A, we describe the construction and basic properties of the lax symmetric monoidal *ℓ-adic realization functors*

$$r_\ell : \mathbf{dgCat}_A \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}\,,\ \ r_\ell^B : \mathbf{dgCat}_B \longrightarrow r_\ell(B) - \mathbf{Mod}$$

such that the following square commutes up to a natural equivalence

$$\begin{array}{ccc} \mathbf{dgCat}_B & \xrightarrow{r_\ell^B} & r_\ell(B) - \mathbf{Mod} \\ \downarrow & & \downarrow \\ \mathbf{dgCat}_A & \xrightarrow[r_\ell]{} & \mathbb{Q}_\ell(\beta) - \mathbf{Mod}. \end{array}$$

Here $\mathbb{Q}_\ell(\beta) := \oplus_{n\in\mathbb{Z}}(\mathbb{Q}_\ell[2](1))^{\otimes^n}$ is viewed as an $\mathbb{E}_\infty$-ring object in the symmetric monoidal ∞-category $\mathsf{L}(S_{\acute{e}t}, \ell)$ of ℓ-adic étale sheaves on $S = \mathbf{Spec}\, A$, and $\mathbf{dgCat}_A$ (resp. $\mathbf{dgCat}_B$) denotes the ∞-category of dg-categories over A (resp. over B, Definition 1.1.2).
Building on these results, we construct a *Chern character*

$$Ch : \mathsf{HK} \to |r_\ell(-)|$$

as a natural lax symmetric monoidal transformation between ∞-functors from $\mathbf{dgCat}_B$ to the ∞-category $\mathbf{Sp}$ of spectra. Here HK is the non-connective homotopy invariant algebraic K-theory of dg-categories over B ([Ro, BRTV]), and, for any $T \in \mathbf{dgCat}_B$, $|r_\ell(T)|$ denotes the spectrum associated to the the hyper-cohomology of $r_\ell(T)$ via the (spectral) Eilenberg-Mac Lane construction.
With the Chern character at our disposal, we prove an *ℓ-adic trace formula* in the following setting. Let T be a smooth, proper dg-category over B, and $f : T \in T$ and endomorphism of T, whose associated perfect $T^o \otimes_B T$-module is denoted by Γ_f. If T satisfies a certain tensorial admissibility property with respect to the ℓ -adic realization r_ℓ (called $r_\ell^\otimes$-admissibility in the text, see Definition 1.3.1), to the effect that the natural coevaluation map exhibits $r_\ell(T^o)$ as the dual of $r_\ell(T)$ as an $r_\ell(B)$-module, then there is a well defined *trace* $\mathrm{Tr}(f_{|r_\ell(T)}) \in \pi_0(|r_\ell(B)|)$ of the endomorphism f. Our ℓ-adic trace formula then states that

$$\mathrm{Tr}(f_{|r_\ell(T)}) = Ch([\mathsf{HH}(T/B, \Gamma_f)]),$$

where $[\mathsf{HH}(T/B, \Gamma_f)]$ is the class in $\mathsf{HK}_0(B)$ of the Hochschild homology of Γ_f (which is indeed a perfect B-dg module).
With an eye to perspective applications, we also establish a version of this ℓ-adic trace formula when the base B is only supposed to be an $\mathbb{E}_2$-algebra (Section 1.4).

We conclude this note by describing a strategy to prove Bloch's conductor conjecture ([Bl], [Ka-Sa, Conj. 6.2.1]), by using our ℓ-adic trace formula and the main results of [BRTV]. Here $S = \mathbf{Spec}A$ is a henselian trait, with perfect residue field k, and fraction field K, and X is a regular scheme, equipped with a proper and flat map to S, such that the generic fiber X_K is smooth over K. We fix a uniformizer π of A, and regard it as a global function on X. Our strategy is divided in two steps. In the first one we explain how to obtain a conductor formula, i.e. an expression of the Artin conductor of X/S in terms of the Euler characteristic of the Hochschild homology of matrix factorizations of the pair (X, π). This conductor formula is interesting in itself, and seems to be new. The second step then consists in identifying our conductor formula with Bloch's.

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Notations. Except where otherwise specificated, we will write $S = \mathbf{Spec}\, A$ for A an excellent commutative ring. We denote by $\mathbf{Sp}$ the ∞-category of spectra, and by $\mathbb{S} \in \mathbf{Sp}$ the sphere spectrum. All functors (e.g. tensor products) will be implicitly derived.

1 ℓ-adic realizations of nc-spaces

1.1 Reminders on dg-categories and their realizations

We consider the category $dgCat_A$ of small A-linear dg-categories and A-linear dg-functors. We remind that an A-linear dg-functor $T \longrightarrow T'$ is a Morita equivalence if the induced functor of the corresponding derived categories of dg-modules $f^* : D(T') \longrightarrow D(T)$ is an equivalence of categories (see [To1] for details). The ∞-category of dg-categories over S is defined to be the localisation of $dgCat_A$ along these Morita equivalences, and will be denoted by $\mathbf{dgCat}_S$ or $\mathbf{dgCat}_A$.

As in [To1, § 4], the tensor product of A-linear dg-categories can be derived to a symmetric monoidal structure on the ∞-category $\mathbf{dgCat}_A$. It is a well known fact that dualizable objects in $\mathbf{dgCat}_S$ are precisely smooth and proper dg-categories over A (see [To2, Prop. 2.5]).

The compact objects in $\mathbf{dgCat}_S$ are the dg-categories of finite type over A in the sense of [To-Va]. We denote their full sub-∞-category by $\mathbf{dgCat}_S^{ft} \subset \mathbf{dgCat}_S$. The full sub-category $\mathbf{dgCat}_S^{ft}$ is preserved by the monoidal structure, and moreover any dg-category is a filtered colimit of dg-categories of finite type. We thus have a natural equivalence of symmetric monoidal ∞-categories

$$\mathbf{dgCat}_S \simeq \mathbf{Ind}(\mathbf{dgCat}_S^{ft}).$$

We denote by $\mathcal{SH}_S$ the stable $\mathbb{A}^1$-homotopy ∞-category of schemes over S (see [Vo, Def. 5.7] and [Ro, § 2]). It is a presentable symmetric monoidal ∞-category whose monoidal structure will be denoted by $\wedge_S$. Homotopy invariant algebraic K-theory defines an $\mathbb{E}_\infty$-ring object in $\mathcal{SH}_S$ that we denote by $\mathbb{BU}_S$ (a more standard notation is KGL). We denote by $\mathbb{BU}_S - \mathbf{Mod}$ the ∞-category of $\mathbb{BU}_S$-modules in $\mathcal{SH}_S$. It is a presentable symmetric monoidal ∞-category whose monoidal structure will be denoted by $\wedge_{\mathbb{BU}_S}$.

As explained in [BRTV], there exists a lax symmetric monoidal ∞-functor

$$\mathsf{M}^- : \mathbf{dgCat}_S \longrightarrow \mathbb{BU}_S - \mathbf{Mod},$$

which is denoted by $T \mapsto \mathsf{M}^T$. The precise construction of the ∞-functor M^- is rather involved and uses in an essential manner the theory of non-commutative motives of [Ro] as well as the comparison with the stable homotopy theory of schemes. Intuitively, the ∞-functor M^- sends a dg-category T to the K-theory functor $S' \mapsto \mathsf{HK}(S' \otimes_S T)$. To be more precise, there is a forgetful ∞-functor

$$\mathsf{U} : \mathbb{BU}_S - \mathbf{Mod} \longrightarrow \mathbf{Fun}^\infty(Sm_S^{op}, \mathbf{Sp}),$$

to the ∞-category of presheaves of spectra on the category Sm_S of smooth S-schemes. For a given dg-category T over S, the presheaf $\mathsf{U}(\mathsf{M}^T)$ is defined by sending a smooth S-scheme $S' = \mathbf{Spec}\, A' \to$

Spec $A = S$ to $\mathsf{HK}(A' \otimes_A T)$, the homotopy invariant non-connective K-theory spetrum of $A' \otimes_A T$ (see [Ro, 4.2.3]).

The ∞-functor M^- satisfies some basic properties which we recall here.

1. The ∞-functor M^- is a localizing invariant, i.e. for any short exact sequence $T_0 \hookrightarrow T \longrightarrow T/T_0$ of dg-categories over A, the induced sequence

$$\mathsf{M}^{T_0} \longrightarrow \mathsf{M}^T \longrightarrow \mathsf{M}^{T/T_0}$$

   exhibits M^{T_0} has the fiber of the morphism $\mathsf{M}^T \to \mathsf{M}^{T/T_0}$ in $\mathbb{BU}_S - \mathbf{Mod}$.

2. The natural morphism $\mathbb{BU}_S \longrightarrow \mathsf{M}^A$, induced by the lax monoidal structure of M^-, is an equivalence of $\mathbb{BU}_S$-modules.

3. The ∞-functor $T \mapsto \mathsf{M}^T$ commutes with filtered colimits.

4. For any quasi-compact and quasi-separated schemes $p : X \longrightarrow S$, we have a natural equivalence of $\mathbb{BU}_S$-modules

$$\mathsf{M}^{\mathsf{L}_{\mathrm{Perf}}(X)} \simeq p_*(\mathbb{BU}_X),$$

   where $p_* : \mathbb{BU}_X - \mathbf{Mod} \longrightarrow \mathbb{BU}_S - \mathbf{Mod}$ is the direct image of $\mathbb{BU}$-modules, and $\mathsf{L}_{\mathrm{Perf}}(X)$ is the dg-category of perfect complexes on X.

We now let ℓ be a prime number invertible in A. We denote by $\mathsf{L}_{\mathsf{ct}}(S_{\acute{e}t}, \ell)$ the ∞-category of constructible $\mathbb{Q}_\ell$-complexes on S. It is a symmetric monoidal ∞-category, and we denote by

$$\mathsf{L}(S_{\acute{e}t}, \ell) := \mathbf{Ind}(\mathsf{L}_{\mathsf{ct}}(S_{\acute{e}t}, \ell))$$

its completion by filtered colimits (see [Ga-Lu, Def. 4.3.26]). According to [Ro, Cor. 2.3.9], there exists an ℓ-adic realization ∞-functor $r_\ell : \mathcal{SH}_S \longrightarrow \mathsf{L}(S_{\acute{e}t}, \ell)$. By construction, r_ℓ is a symmetric monoidal ∞-functor sending a smooth scheme $p : X \longrightarrow S$ to $p_\sharp p^!(\mathbb{Q}_\ell)$, or, in other words, to the relative ℓ-adic homology of X over S.

We let $\mathbb{T} := \mathbb{Q}_\ell[2](1)$, and we consider the $\mathbb{E}_\infty$-ring object in $\mathsf{L}(S_{\acute{e}t}, \ell)$

$$\mathbb{Q}_\ell(\beta) := \oplus_{n \in \mathbb{Z}} \mathbb{T}^{\otimes n}.$$

In this notation, β stands for $\mathbb{T}$, and $\mathbb{Q}_\ell(\beta)$ for the algebra of Laurent polynomials in β, so we could also have written

$$\mathbb{Q}_\ell(\beta) = \mathbb{Q}_\ell[\beta, \beta^{-1}].$$

As shown in [BRTV], there exists a canonical equivalence of $\mathbb{E}_\infty$-ring objects in $\mathsf{L}(S_{\acute{e}t}, \ell)$ $r_\ell(\mathbb{BU}_S) \simeq \mathbb{Q}_\ell(\beta)$, which is induced by the Chern character from algebraic K-theory to étale cohomology. We thus obtain a symmetric monoidal ∞-functor

$$r_\ell : \mathbb{BU}_S - \mathbf{Mod} \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod},$$

from $\mathbb{BU}_S$-modules in $\mathcal{SH}_S$ to $\mathbb{Q}_\ell(\beta)$-modules in $\mathsf{L}(S_{\acute{e}t}, \ell)$. Composed with the construction $T \mapsto \mathsf{M}^T$ we obtain a lax monoidal ∞-functor

$$r_\ell : \mathbf{dgCat}_S \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}.$$

Definition 1.1.1 *The ∞-functor defined above*

$$r_\ell : \mathbf{dgCat}_S \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}$$

is called the ℓ-adic realization functor for dg-categories.

From the standard properties of the functor $T \mapsto \mathsf{M}^T$, recalled above, we obtain the following properties for the ℓ-adic realization functor $T \mapsto r_\ell(T)$.

1. The ∞-functor r_ℓ is a localizing invariant, i.e. for any short exact sequence $T_0 \hookrightarrow T \longrightarrow T/T_0$ of dg-categories over A, the induced sequence

   $$r_\ell(T_0) \longrightarrow r_\ell(T) \longrightarrow r_\ell(T/T_0)$$

   is a fibration sequence in $\mathbb{Q}_\ell(\beta) - \mathbf{Mod}$.

2. The natural morphism

   $$\mathbb{Q}_\ell(\beta) \longrightarrow r_\ell(A),$$

   induced by the lax monoidal structure, is an equivalence in $\mathbb{Q}_\ell(\beta) - \mathbf{Mod}$.

3. The ∞-functor r_ℓ commutes with filtered colimits.

4. For any separated morphism of finite type $p : X \longrightarrow S$, we have a natural morphism of $\mathbb{Q}_\ell(\beta)$-modules

   $$r_\ell(\mathsf{L}_{\mathrm{Perf}}(X)) \longrightarrow p_*(\mathbb{Q}_\ell(\beta)),$$

   where $p_* : \mathbb{Q}_\ell(\beta) - \mathbf{Mod} \longrightarrow \mathbb{Q}_\ell(\beta) - \mathbf{Mod}$ is induced by the direct image $L_{\mathsf{ct}}(X_{\acute{e}t}, \ell) \longrightarrow L_{\mathsf{ct}}(S_{\acute{e}t}, \ell)$ of constructible $\mathbb{Q}_\ell$-complexes. If p is proper, or if A is a field, this morphism is an equivalence.

To finish this part, we generalize a bit the above setting, by adding a base $\mathbb{E}_\infty$-algebra B, and considering B-linear dg-categories.

Let B be an $\mathbb{E}_\infty$-algebra over A. We consider B as an $\mathbb{E}_\infty$-monoid in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$. We define dg-categories over B as being B-modules in $\mathbf{dgCat}_A$. More specifically we have the following notion.

Definition 1.1.2 *The* symmetric monoidal ∞-category of (small) B-linear dg-categories *is defined to be the ∞-category of B-modules in* $\mathbf{dgCat}_A$*. It is denoted by*

$$\mathbf{dgCat}_B := B - \mathbf{Mod}_{\mathbf{dgCat}_A},$$

and his monoidal structure by $\otimes_B$*.*

Applying our ℓ-adic realization functor (Definition 1.1.1), we have that $r_\ell(B)$ is an $\mathbb{E}_\infty$ $\mathbb{Q}_\ell(\beta)$-algebra. This induces a lax symmetric monoidal ∞-functor

$$r_\ell^B : \mathbf{dgCat}_B \longrightarrow r_\ell(B) - \mathbf{Mod}.$$

By construction, the natural forgetful ∞-functors makes the following square commutes up to a natural equivalence

$$\begin{array}{ccc} \mathbf{dgCat}_B & \xrightarrow{r_\ell^B} & r_\ell(B) - \mathbf{Mod} \\ \downarrow & & \downarrow \\ \mathbf{dgCat}_A & \xrightarrow[r_\ell]{} & \mathbb{Q}_\ell(\beta) - \mathbf{Mod}. \end{array}$$

We will often consider B as implicitely given and write r_ℓ for r_ℓ^B.

1.2 Chern character and Grothendieck-Riemann-Roch

We fix as above an $\mathbb{E}_\infty$-algebra B over A. As explained in [BRTV], there is a symmetric monoidal ∞-category $\mathcal{SH}_B^{nc}$ of non-commutative motives over B. As an ∞-category it is the full sub-∞-category of ∞-functors of (co)presheaves of spectra

$$\mathbf{dgCat}_B^{ft} \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance. The symmetric monoidal structure is induced by left Kan extension of the symmetric monoidal structure $\otimes_B$ on $\mathbf{dgCat}_B^{ft}$.

We consider $\Gamma : \mathsf{L}(S_{\acute{e}t}, \ell) \longrightarrow \mathbf{dg}_{\mathbb{Q}_\ell}$, the global section ∞-functor, taking an ℓ-adic complex on $S_{\acute{e}t}$ to its hyper-cohomology. Composing this with the Dold-Kan construction $\mathbb{R}\mathbf{Map}_{\mathbf{dg}_{\mathbb{Q}_\ell}}(\mathbb{Q}_\ell, -) : \mathbf{dg}_{\mathbb{Q}_\ell} \longrightarrow \mathbf{Sp}$, we obtain an ∞-functor

$$|-| : \mathsf{L}(S_{\acute{e}t}, \ell) \longrightarrow \mathbf{Sp},$$

which morally computes hyper-cohomology of $S_{\acute{e}t}$ with ℓ-adic coefficients, i.e. for any $E \in \mathsf{L}(S_{\acute{e}t}, \ell)$, we have natural isomorphisms

$$H^i(S_{\acute{e}t}, E) \simeq \pi_{-i}(|E|), i \in \mathbb{Z}.$$

By what we have seen in our last paragraph, the composite functor $T \mapsto |r_\ell(T)|$ provides a (co)presheaves of spectra

$$\mathbf{dgCat}_B^{ft} \longrightarrow \mathbf{Sp},$$

satisfying Nisnevich descent and $\mathbb{A}^1$-homotopy invariance. It thus defines an object $|r_\ell| \in \mathcal{SH}_B^{nc}$. The fact that r_ℓ is lax symmetric monoidal implies moreover that $|r_\ell|$ is endowed with a natural structure of a $\mathbb{E}_\infty$-ring object in $\mathcal{SH}_B^{nc}$.

Each $T \in \mathbf{dgCat}_B^{ft}$ defines a corepresentable object $h^T \in \mathcal{SH}_B^{nc}$, characterized by the ($\infty$-)functorial equivalences

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_B^{nc}}(h^T, F) \simeq F(T),$$

for any $F \in \mathcal{SH}_B^{nc}$. The existence of h^T is a formal statement, however the main theorem of [Ro] implies that we have natural equivalences of spectra

$$\mathbb{R}\mathbf{Map}_{\mathcal{SH}_B^{nc}}(h^T, h^B) \simeq \mathsf{HK}(T),$$

where $\mathsf{HK}(T)$ stands for non-connective homotopy algebraic K-theory of the dg-category T. In other words, $T \mapsto \mathsf{HK}(T)$ defines an object in $\mathcal{SH}_B^{nc}$ which is isomorphic to h^B. By Yoneda lemma, we thus obtain an equivalence of spaces

$$\mathbb{R}\mathbf{Map}^{lax-\otimes}(\mathsf{HK}, |r_\ell|) \simeq \mathbb{R}\mathbf{Map}_{\mathbb{E}_\infty - \mathbf{Sp}}(\mathbb{S}, |r_\ell(B)|) \simeq *.$$

In other words, there exists a unique lax symmetric monoidal natural transformation

$$\mathsf{HK} \longrightarrow |r_\ell|,$$

between lax monoidal ∞-functors from $\mathbf{dgCat}_B^{ft}$ to $\mathbf{Sp}$. We extend this to all dg-categories over B as usual by passing to Ind-completion $\mathbf{dgCat}_B \simeq \mathbf{Ind}(\mathbf{dgCat}_B^{ft})$.

Definition 1.2.1 *The natural transformation defined above is called the* ℓ-adic Chern character. *It is denoted by*

$$Ch : \mathsf{HK}(-) \longrightarrow |r_\ell(-)|.$$

Note that the Chern character is an absolute construction, and does not depend on the base B. In other words, it factors through the natural forgetful functor $\mathbf{dgCat}_B \longrightarrow \mathbf{dgCat}_A$. There is thus no need to specify the base B in the notation Ch.

Note that Definition 1.2.1 contains a formal Grothendieck-Riemann-Roch formula. Indeed, for any B-linear dg-functor $f : T \longrightarrow T'$, the square of spectra

$$\begin{array}{ccc} \mathsf{HK}(T) & \xrightarrow{f_!} & \mathsf{HK}(T') \\ {\scriptstyle Ch_T}\downarrow & & \downarrow{\scriptstyle Ch_{T'}} \\ r_\ell(T) & \xrightarrow[f_!]{} & r_\ell(T') \end{array}$$

commutes up to a natural equivalence.

1.3 The trace formula

In the last paragraph of this section we prove an ℓ-adic trace formula for smooth and proper dg-categories over some base $\mathbb{E}_\infty$-algebra B. The formula is a direct consequence of the dualizability of smooth and proper dg-categories, and the functoriality of the Chern character.

We let B be a fixed $\mathbb{E}_\infty$-algebra over A and let $\mathbf{dgCat}_B$ the symmetric monoidal ∞-category of dg-categories over B. Recall that the dualizable objects T in $\mathbf{dgCat}_B$ are precisely the smooth and proper dg-categories (see [To2, Prop. 2.5]). For such dg-categories T, the dual $T^\vee$ in $\mathbf{dgCat}_B$ is exactly the opposite dg-category T^o, and the evaluation map

$$T^o \otimes_B T \longrightarrow B$$

is simply given by sending (x, y) to the perfect B-module $T(x, y)$. In the same manner, the coevaluation map

$$B \longrightarrow T^o \otimes_B T$$

is given the identity bi-module T. In particular, the composition

$$B \longrightarrow T^o \otimes_B T \longrightarrow B$$

is the endomorphism of B given by the perfect B-dg module $\mathsf{HH}(T/B)$, the Hochschild chain complex of T over B.

Definition 1.3.1 *We say that a smooth and proper dg-category T over B is* $r_\ell^\otimes$-admissible *if the natural morphism, induced by the lax monoidal structure*

$$r_\ell(T^o) \otimes_{r_\ell(B)} r_\ell(T) \longrightarrow r_\ell(T^o \otimes_B T)$$

is an equivalence in $\mathsf{L}(S_{\acute{e}t}, \ell)$.

A direct observation shows that if $T \in \mathbf{dgCat}_B$ is smooth, proper and $r_\ell^\otimes$-*admissible*, then the image by r_ℓ of the co-evaluation map

$$r_\ell(B) \longrightarrow r_\ell(T^o \otimes_B T) \simeq r_\ell(T^o) \otimes_{r_\ell(B)} r_\ell(T)$$

exhibits $r_\ell(T^o)$ as the dual of $r_\ell(T)$ as a $r_\ell(B)$-module.

We now fix a smooth, proper and $r_\ell^\otimes$-admissible dg-category over B. Let $f : T \longrightarrow T$ be an endomorphism of T in $\mathbf{dgCat}_B$. By duality, it corresponds to a a perfect $T^o \otimes_B T$-dgmodule Γ_f. The Hochschild homology of Γ_f is the perfect B-dg module $\mathsf{HH}(T/B, \Gamma_f)$, defined as the composite morphism in $\mathbf{dgCat}_B$

$$B \xrightarrow{\Gamma_f} T^o \otimes_B T \xrightarrow{\mathsf{ev}} B.$$

It thus defines a class in $\mathsf{HK}_0(B)$

$$[\mathsf{HH}(T/B, \Gamma_f)] \in \mathsf{HK}_0(B).$$

On the other hand, the endomorphism induces an endomorphism of $r_\ell(B)$-modules $f : r_\ell(T) \longrightarrow r_\ell(T)$. The trace of this endomorphism is again defined as the composite

$$r_\ell(B) \xrightarrow{\Gamma_f} r_\ell(T)^\vee \otimes_{r_\ell(B)} r_\ell(T) \xrightarrow{\mathsf{ev}} r_\ell(B).$$

The equivalence class of this morphism is identified with an element

$$\mathrm{Tr}(f_{|r_\ell(T)}) \in \pi_0(|r_\ell(B)|).$$

Our ℓ-adic trace formula, which is a direct consequence of duality in $\mathbf{dgCat}_B$ and in $r_\ell(B) - \mathbf{Mod}$, is the following statement

Theorem 1.3.2 *Let $T \in \mathbf{dgCat}_B$ be a smooth, proper and $r_\ell^\otimes$-admissible dg-category over B, and $f : T \longrightarrow T$ be an endomorphism. Then we have*

$$Ch([\mathsf{HH}(T/B, \Gamma_f)]) = \mathrm{Tr}(f_{|r_\ell(T)}).$$

A case of special interests is when B is such that the natural morphisms of rings

$$\mathbb{Z} \longrightarrow \mathsf{HK}_0(B) \qquad \mathbb{Q}_\ell \longrightarrow \pi_0(|r_\ell(B)|)$$

are both isomorphisms. In this case the Chern character $Ch : \mathsf{HK}_0(B) \longrightarrow H^0(|r_\ell(B)|)$ is the natural inclusion $\mathbb{Z} \subset \mathbb{Q}_\ell$, and the formula reads as an equality of ℓ-adic numbers

$$[\mathsf{HH}(T/B, \Gamma_f)] = \mathrm{Tr}(f_{|r_\ell(T)}).$$

The left hand side of this formula should be interpreted as the intersection number of the graph Γ_f with the diagonal of T, and thus as a virtual number of fixed points of f.

1.4 Extension to $\mathbb{E}_2$-bases

The trace formula of the last section can be extended to the case where the base $\mathbb{E}_\infty$-algebra B is only assumed to be an $\mathbb{E}_2$-algebra. We will here briefly mention how this works.

Let B an $\mathbb{E}_2$-algebra over A. It can be considered as an $\mathbb{E}_1$-algebra object in the symmetric monoidal ∞-category $\mathbf{dgCat}_A$. We thus define the ∞-category of dg-categories enriched over B as

$$\mathbf{dgCat}_B := B - \mathbf{Mod},$$

the ∞-category of B-modules in $\mathbf{dgCat}_A$. As opposed to the $\mathbb{E}_\infty$-case $\mathbf{dgCat}_B$ is no more a monoidal ∞-category, and dualizability must therefore be understood in a slightly different manner.

We will for a while work in $\mathbf{dgCat}_A^{big}$, en enlargement of $\mathbf{dgCat}_A$ by allowing arbirtary bimodules as morphisms. More precisely, $\mathbf{dgCat}_A^{big}$ is an ∞-category whose objects are small dg-categories over A. Morphisms between T and T' in $\mathbf{dgCat}_A^{big}$ are given by the space of all $T \otimes_A (T')^o$-dgmodules. Put differently, $\mathbf{dgCat}_A^{big}$ is the ∞-category of all compactly generated presentable dg-categories over A and continuous dg-functors. The tensor product of dg-categories endows $\mathbf{dgCat}_A^{big}$ with a symmetric monoidal structure still denoted by $\otimes_A$. We have a natural symmetric monoidal ∞-functor $\mathbf{dgCat}_A \longrightarrow \mathbf{dgCat}_A^{big}$, which identifies $\mathbf{dgCat}_A$ with the sub-∞-category of continuous morphisms preserving compact objects. As these two ∞-categories only differ by their morphisms, we will use the expression *big morphisms* to mean morphisms in $\mathbf{dgCat}_A^{big}$.

Back to our $\mathbb{E}_2$-algebra B, the diagonal B-module provide a big morphism of dg-categories

$$B \otimes_A B^o \longrightarrow A.$$

As B is an $\mathbb{E}_2$-algebra, this morphism is moreover a (lax) morphism of $\mathbb{E}_1$-algebra objects in $\mathbf{dgCat}_A^{big}$. The induced base change on the ∞-categories of modules (in $\mathbf{dgCat}_A^{big}$) will be denoted by

$$- \otimes_{B\otimes_A B^o} B : B \otimes_A B^o - \mathbf{Mod} \longrightarrow A - \mathbf{Mod} = \mathbf{dgCat}_A^{big}.$$

Therefore, for any $T \in \mathbf{dgCat}_B$, we can define $T \otimes_B T^o$ by the formula

$$T \otimes_B T^o := (T \otimes_A T^o) \otimes_{B\otimes_A B^o} B \in \mathbf{dgCat}_A.$$

The B-module structure on T

$$B \otimes_A T \longrightarrow T$$

provides a big morphism

$$\mathsf{ev} : T \otimes_A T^o \longrightarrow B^o,$$

which is a morphism of $B \otimes_A B^o$-modules. By base change from $B \otimes_A B^o$ we get an induced evaluation morphism in $\mathbf{dgCat}_A^{big}$

$$\mathsf{ev}' : T \otimes_B T^o \longrightarrow B \otimes_{B\otimes_A B^o} B^o =: \mathsf{HH}(B/A).$$

Note that ev' is only a big morphism of A-linear dg-categories, there is no natural B-linear structure on $T \otimes_B T^o$. Note moreover that as B is an $\mathbb{E}_2$-algebra $\mathsf{HH}(B/A) \simeq S^1 \otimes_A B$ is itself an $\mathbb{E}_1$-algebra and thus can be considered as an object of $\mathbf{dgCat}_A$. We leave to the reader the dual construction of a coevaluation big morphism

$$\mathsf{coev} : A \longrightarrow T^o \otimes_B T.$$

More generally, if $f : T \longrightarrow T$ is an endomorphism of T in $\mathbf{dgCat}_B$, we have a graph morphism in $\mathbf{dgCat}_A^{big}$

$$\Gamma_f : A \longrightarrow T^o \otimes_B T.$$

Definition 1.4.1 *1. A B-linear dg-category $T \in \mathbf{dgCat}_B$ is* smooth (resp. proper) *if the coevaluation* coev *(resp. evaluation* ev*) big morphism lies in $\mathbf{dgCat}_A$ (i.e. preserves compact objects).*

2. For a smooth and proper B-linear dg-category T with and endomorphism $f : T \to T$ in $\mathbf{dgCat}_B$, the $\mathbb{E}_2$-Hochschild homology of T relative to B with coefficients in f *is defined to be the composite morphism in $\mathbf{dgCat}_A$*

$$\mathsf{HH}^{\mathbb{E}_2}(T/B, f) := \; A \xrightarrow{\mathsf{coev}} T \otimes_B T^o \xrightarrow{\mathsf{ev}'} \mathsf{HH}(B/A).$$

Note that by definition the Hochschild homology of the pair (T, f) can be identified with a perfect $\mathsf{HH}(B/A)$-module. As B is an $\mathbb{E}_2$-algebra the natural augmentation $\mathsf{HH}(B/A) \longrightarrow B$ is an $\mathbb{E}_1$-morphism. We thus can base change $\mathsf{HH}(T/B, f)$ defined above to get a perfect B-dgmodule $\mathsf{HH}^{\mathbb{E}_2}(T/B, f) \otimes_{\mathsf{HH}(B/A)} B$. When B is an $\mathbb{E}_\infty$-algebra the Hoschchild homology and $\mathbb{E}_2$-Hochschild homology are related by a natural equivalence of B-dg-modules

$$\mathsf{HH}(T/B, f) \simeq \mathsf{HH}^{\mathbb{E}_2}(T/B, f) \otimes_{\mathsf{HH}(B/A)} B.$$

To be more precise, we have an equivalence of perfect $\mathsf{HH}(B/A)$-dg-modules

$$\mathsf{HH}^{\mathbb{E}_2}(T/B, f) \simeq \mathsf{HH}(T/A, f).$$

The trace formula is the $\mathbb{E}_2$-setting now reads as follows. We fix an $\mathbb{E}_2$-algebra B and a smooth and proper B-linear dg-category T endowed with a B-linear endormorphism $f : T \longrightarrow T$.
As in Definition 1.3.1, we make the following

- **$r_\ell^\otimes$-admissibility assumptions.** The following natural morphisms

$$r_\ell(T) \otimes_{r_\ell(B)} r_\ell(T^o) \longrightarrow r_\ell(T \otimes_B T^o), \qquad \mathsf{HH}(r_\ell(B)/\mathbb{Q}_\ell(\beta)) \longrightarrow r_\ell(\mathsf{HH}(B/A)) \tag{Adm}$$

  are equivalences in $\mathsf{L}(S_et, \ell)$.

Under these assumptions the evaluation and coevaluation morphisms induce a well defined morphisms

$$\mathsf{coev} : \mathbb{Q}_\ell(\beta) \longrightarrow r_\ell(T) \otimes_{r_\ell(B)} r_\ell(T^o) \qquad \mathsf{ev} : r_\ell(T) \otimes_{\mathbb{Q}_\ell(\beta)} r_\ell(T^o) \longrightarrow r_\ell(B).$$

Here the morphism coev is $\mathbb{Q}_\ell(\beta)$-linear and ev is $r_\ell(B) \otimes_{\mathbb{Q}_\ell(\beta)} r_\ell(B^o)$-linear. Theses two morphisms exhibits $r_\ell(T^o)$ has the right dual of $r_\ell(T)$ in $r_\ell(B)$-modules. In particular, the endomorphism f and the evaluation morphism provides a well defined trace morphism

$$\mathrm{Tr}(f) : \mathbb{Q}_\ell(\beta) \longrightarrow r_\ell(B) \otimes_{r_\ell(B) \otimes_{\mathbb{Q}_\ell(\beta)} r_\ell(B^o)} r_\ell(B^o),$$

and thus a well defined element

$$\mathrm{Tr}(f_{|r_\ell(T)}) \in H^0(S_{\acute{e}t}, \mathsf{HH}(r_\ell(B)/\mathbb{Q}_\ell(\beta))).$$

The $\mathbb{E}_2$-version of the trace formula of Theorem 1.3.2 is the following

Theorem 1.4.2 *With the above notations and assumptions we have an equality*

$$Ch([\mathsf{HH}^{\mathbb{E}_2}(T/B, f)]) = \mathrm{Tr}(f_{|r_\ell(T)})$$

in $H^0(S_{\acute{e}t}, \mathsf{HH}(r_\ell(B)/\mathbb{Q}_\ell(\beta)))$.

By using the augmentation $\rho : \mathsf{HH}(B/A) \longrightarrow B$ we deduce another trace formula in a more familiar form

$$Ch([\mathsf{HH}^{\mathbb{E}_2}(T/B, f) \otimes_{\mathsf{HH}(B/A)} B]) = \rho(\mathrm{Tr}(f_{|r_\ell(T)})),$$

which is now an equality in $H^0(S_{\acute{e}t}, r_\ell(B))$.

2 Applications to vanishing cycles and Bloch's conductor conjecture

Our trace formula (Theorem 1.3.2 and Theorem 1.4.2), combined with the main result of [BRTV], have some interesting applications to Bloch's conductor conjecture of [Bl].

2.1 Bloch's conductor conjecture

Our base scheme is an henselian trait $S = \mathbf{Spec}\, A$, with perfect residue field k, and fraction field K. Let $X \longrightarrow S$ be proper and flat morphism of finite type, and of relative dimension n. We assume that the generic fiber X_K is smooth over K, and that X is a regular scheme. Bloch's conductor formula conjecture reads as follows (see [Bl, Ka-Sa], for detailed definitions of the various objects involved).

Conjecture 2.1.1 [Bloch's conductor Conjecture] *We have an equality*

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell) - Sw(X_{\bar{K}}),$$

where $\chi(Y, \ell)$ *denotes the* ℓ*-adic Euler characteristic of a variety* Y *for* ℓ *prime to the characteristic of* k*,* $Sw(X_{\bar{K}})$ *is the Swan conductor of the* $Gal(K)$*-representation* $H^*(X_{\bar{K}}, \mathbb{Q}_\ell)$*, and* $[\Delta_X, \Delta_X]_S$ *is the degree in* $CH_0(k) \simeq \mathbb{Z}$ *of Bloch's localised self-intersection* $(\Delta_X, \Delta_X)_S \in \mathrm{CH}_0(X_k)$ *of the diagonal in* X*. The (negative of the) rhs is called the Artin conductor of* X/S*.*

Note that this conjecture is known in some important cases: it is classical for $n = 0$, proved by Bloch himself for $n = 1$ [Bl], and for arbitrary n by Kato and Saito [Ka-Sa] under the hypothesis that the reduced special fiber $(X_k)_{\mathrm{red}}$ is a normal crossing divisor. By [Or], we also know that Conjecture 2.1.1 implies the Deligne-Milnor Conjecture for isolated singularities (both in the equi and mixed characteristic case). The isolated singularities and equicharacteristic case of the Deligne-Milnor conjecture was proved by Deligne [SGA7-II, Exp. XVI], while the isolated singularities and mixed characteristic case was proved by Orgogozo [Or] for relative dimension $n = 1$.

2.2 A strategy

We propose here an approach to Bloch's conjecture based on our trace formula (Theorem 1.4.2) and the use of matrix factorizations as developed in [BRTV]. Below we will just sketch a general strategy that might lead to a proof of Conjecture 2.1.1. Some details are still to be fixed, as for instance the comparison formula (Comp) with Bloch's localised self-intersection of the diagonal. These details are

currently being checked and, hopefully, they will appear in a forthcoming paper.

Let π be a uniformizer for A, so that $k = A/\pi$. We let $T := \mathsf{MF}(X, \pi)$ be the A-linear dg-category of matrix factorizations of X for the function π on X, as studied in detail in [BRTV]. We consider the $\mathbb{E}_2$-algebra B^+ defined as

$$B^+ := \mathbb{R}\underline{Hom}_{k\otimes_A k}(k, k).$$

The fact that B^+ is an $\mathbb{E}_2$-algebra follows formally from the fact that it is endowed with two compatible $\mathbb{E}_1$-multiplications, one given by composition of endomorphisms, and another coming from the fact that the derived scheme $\mathbf{Spec}\,(k \otimes^L_A k)$ is actually a Segal groupoid object in derived schemes: the descent groupoid of the map $\mathbf{Spec}\, k \to S$. The composition law in this groupoid induces another $\mathbb{E}_1$-structure on B^+ given by convolution, making it into an $\mathbb{E}_2$-algebra over A.

As an $\mathbb{E}_1$-algebra, B^+ can be identified with $k[u]$, where u is a free variable in degree 2. We set

$$B := B^+[u^{-1}].$$

Again, as an $\mathbb{E}_1$-algebra B is $k[u, u^{-1}]$. However, in general, B is not equivalent to $k[u, u^{-1}]$ as an $\mathbb{E}_2$-algebra, as, a priori, it is not even linear over k. It can be shown that if A is a k-algebra, then B is equivalent to $k[u, u^{-1}]$ and thus is indeed an $\mathbb{E}_\infty$-algebra over k.

In [BRTV] it is shown that the $\mathbb{E}_2$-algebra B naturally acts on the dg-category $T = \mathsf{MF}(X, \pi)$ making it into a B-linear dg-category. Note however that we have a natural ring isomorphisms

$$\chi : \mathsf{HK}_0(B) \simeq \mathbb{Z}.$$

A key result is the the following proposition, whose proof will appear elswhere.

Proposition 2.2.1 *With the above notations we have a natural Morita equivalence of A-linear dg-categories*

$$\mathsf{MF}(X, \pi)^o \otimes_B \mathsf{MF}(X, \pi) \otimes_B T^o \simeq L_{sg}(X \times_S X),$$

where the right hand side denotes the quotient $L^b_{coh}(X \times_S X)/L_{parf}(X \times_S X)$ computed in the Morita theory of dg-categories.

In the above result, the diagonal bimodule of $\mathsf{MF}(X, \pi)$ corresponds to the structure sheaf of the diagonal $\Delta_X \in L_{sg}(X \times_S X)$. This easily implies that $\mathsf{MF}(X, \pi)$ is a smooth and proper dg-categories over B. Moreover, the arguments developed in [BRTV] together with proposition 2.2.1 will imply that $\mathsf{MF}(X, \pi)$ satisfies the $r^{\otimes}_\ell$-admissibility conditions when the inertia group $I \subset Gal(K^{sp}/K)$ acts unipotently on $H^*_{et}(X_{\overline{K}}, \mathbb{Q}_\ell)$.

Let us now come to our strategy for a possible proof of the Bloch's Conjecture 2.1.1. It is divided in two steps, and already the first one yields an interesting, and apparently new formula for the Artin conductor.

First step: a conductor formula. We treat first the case of unipotent monodromy.

The unipotent case. We first assume the extra condition that the intertia subgroup $I \subset G_K = Gal(K^{sp}/K)$ acts with unipotent monodromy on all the $\mathbb{Q}_\ell$-spaces $H^i_{\acute{e}t}(X_{\bar{K}}, \mathbb{Q}_\ell)$. This implies that

the monodromy action is in particular tame and that the Swan conductor term vanishes. Thus, Bloch's formula now reads

$$[\Delta_X, \Delta_X]_S = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell).$$

Corollary 2.2.2 *With the unipotent assumption above wee have*

$$\chi(\mathsf{HH}(T/A) \otimes_{\mathsf{HH}(B/A)} B) = \chi(X_{\bar{k}}, \ell) - \chi(X_{\bar{K}}, \ell). \tag{CF-uni}$$

Proof. This is a direct consequence of our trace formula (Theorem 1.4.2), and the main theorem of [BRTV], stating the existence of a natural equivalence of $\mathbb{Q}_\ell(\beta) \oplus \mathbb{Q}_\ell(\beta)[-1](1)$-modules

$$r_\ell(T) \simeq \mathbf{H}_{\acute{e}t}(X_{\bar{k}}, \Phi_{X,\ell}[-1](\beta))^{hI},$$

where $\Phi_{X,\ell}$ is the complex of vanishing cycles on X_k, and $(-)^{hI}$ denotes the homotopy invariants for the action of the inertia group I. □

Remark 2.2.3 (1) Note that the B-linear dg-category $T = \mathsf{MF}(X, \pi)$ is not expected be $H^\otimes$-admissible unless the monodromy action is unipotent. This is directly related to the fact that $r_\ell(T)$ provides hI-invariants vanishing cohomology, and taking invariants in general does not commute with tensor products. However, when I acts unipotently, taking hI-invariants commutes with tensor products when computed over the dg-algebra $\mathbb{Q}_\ell^{hI} = \mathbb{Q}_\ell \oplus \mathbb{Q}_\ell[-1](1)$. This is why Corollary 2.2.2 cannot be true for non-unipotent monodromy.
(2) Note that Bloch's conductor conjecture was open, without conditions on the reduced special fiber, even for unipotent monodromy so corollary 2.2.2 already provides new cases where the conjecture is true.

Extension to the non-unipotent case. By Grothendieck's theorem ([SGA7-I, Exp. I]), the action of the inertia I is always quasi-unipotent. Let $S' \longrightarrow S$ be the totally ramified covering corresponding to a totally ramified finite Galois extension K'/K with group G, such that the base change $X' := X \times_S S' \longrightarrow S'$ has unipotent monodromy. The finite group G acts on $X' \longrightarrow S'$ and induces a morphism of quotient stacks

$$X'_G := [X'/G] \longrightarrow S'_G := [S'/G].$$

We now consider the category T' of matrix factorizations of X'_G relative to π. This category is now linear over some G-equivariant $\mathbb{E}_2$-algebra B_G. The underlying $\mathbb{E}_2$-algebra B_G is now $\mathbb{R}\underline{Hom}_{k\otimes_{A'}k}(k,k)[u^{-1}]$ where $S' = \mathbf{Spec}\, A'$.

As the monodromy for $X' \longrightarrow S'$ is unipotent again, we conjecture that T' is $H^\otimes$-admissible as a G-equivariant B_G-linear dg-category, as well as smooth and proper. This leads to a G-equivariant version of Corollary 2.2.2 giving now a formula for the character of the G-representation on $H^*_{\acute{e}t}(X', \nu[-1])^{hI}(\beta)$ in terms of the G-action on $\mathsf{HH}(T'/B_G)$. We hope that this G-equivariant version will impliy that

$$\chi(\mathsf{HH}(\mathsf{MF}(X,\pi)/B) \otimes_{\mathsf{HH}(B/A)} B) = \chi(X_{\bar{k}}) - \chi(X_{\bar{K}}) - Sw(X_{\bar{K}}). \tag{CF-gen}$$

Second step: our conductor formula coincides with Bloch's conductor formula. To complete the proof of Bloch's conductor Conjecture, we need to compare the localized intersection number of Conjecture 2.1.1 with the Euler characteristic of the B-dg-module $\mathsf{HH}(T/B) \otimes_{\mathsf{HH}(B/A)} B$, with $T = \mathsf{MF}(X, \pi)$. This comparison should be an equality

$$[\Delta_X, \Delta_X]_S = \chi(\mathsf{HH}(T/B) \otimes_{\mathsf{HH}(B/A)} B). \tag{Comp}$$

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