Algebraic ∞-categories, Θ-categories and tannakian duality

Bertrand Toën
joint with J. Nuiten, G. Vezzosi and ChatPPP[1]

CNRS, Université de Toulouse

CATS8, Oléron, September 2026

[1]Old and forgotten model, not quite an LLM but works the same way. It is way cheaper and slow, and makes a lot of mistakes. "PPP" stands for "Piece of Paper and Pen".

Origin of this work

1. Grothendieck: *Pursuing stacks*, letters to Breen (1974) and letter to Verdier (1966 - already mentioning "∞-categories"!).

Origin of this work

1. Grothendieck: *Pursuing stacks*, letters to Breen (1974) and letter to Verdier (1966 - already mentioning "∞-categories"!).
2. C. Simpson's non-abelian Hodge theory: *homotopy over the complex numbers* (1994).

Origin of this work

1. Grothendieck: *Pursuing stacks*, letters to Breen (1974) and letter to Verdier (1966 - already mentioning "∞-categories"!).
2. C. Simpson's non-abelian Hodge theory: *homotopy over the complex numbers* (1994).
3. One of the original motivations for DAG (1999-2000): tannakian formalism in the derived context.

Origin of this work

1. Grothendieck: *Pursuing stacks*, letters to Breen (1974) and letter to Verdier (1966 - already mentioning "∞-categories"!).
2. C. Simpson's non-abelian Hodge theory: *homotopy over the complex numbers* (1994).
3. One of the original motivations for DAG (1999-2000): tannakian formalism in the derived context.
4. L. Katzarkov's question: ($\sim$2000) *What is the homotopical counterpart of the theory of vanishing cycles?*

Some known results

1. Tannakian formalism for 1-categories over fields (Grothendieck-Saavedra-Deligne-Milne). Extension to Dedekind rings (Wedhorn, 2003, Duong, 2013 ...).

Some known results

1. Tannakian formalism for 1-categories over fields (Grothendieck-Saavedra-Deligne-Milne). Extension to Dedekind rings (Wedhorn, 2003, Duong, 2013 ...).
2. Reconstruction of quotient stack (Savin, 2004).

Some known results

1. Tannakian formalism for 1-categories over fields (Grothendieck-Saavedra-Deligne-Milne). Extension to Dedekind rings (Wedhorn, 2003, Duong, 2013 ...).
2. Reconstruction of quotient stack (Savin, 2004).
3. k-linear stable ∞-categories, k field of characteristic 0 (J. Wallbridge 2012, J. Lurie 2011, I. Iwanari 2014).

Some known results

1. Tannakian formalism for 1-categories over fields (Grothendieck-Saavedra-Deligne-Milne). Extension to Dedekind rings (Wedhorn, 2003, Duong, 2013 ...).
2. Reconstruction of quotient stack (Savin, 2004).
3. k-linear stable ∞-categories, k field of characteristic 0 (J. Wallbridge 2012, J. Lurie 2011, I. Iwanari 2014).
4. Special cases over ring spectra (J. Wallbridge 2012).

Some known results

1. Tannakian formalism for 1-categories over fields (Grothendieck-Saavedra-Deligne-Milne). Extension to Dedekind rings (Wedhorn, 2003, Duong, 2013 ...).
2. Reconstruction of quotient stack (Savin, 2004).
3. k-linear stable ∞-categories, k field of characteristic 0 (J. Wallbridge 2012, J. Lurie 2011, I. Iwanari 2014).
4. Special cases over ring spectra (J. Wallbridge 2012).
5. Tannakian reconstruction of spectral schemes and stacks (J. Lurie 2004, Bhatt-Halpern-Leistner 2015, Antieau-Stefanich 2025).

Some known results

1. Tannakian formalism for 1-categories over fields (Grothendieck-Saavedra-Deligne-Milne). Extension to Dedekind rings (Wedhorn, 2003, Duong, 2013 ...).
2. Reconstruction of quotient stack (Savin, 2004).
3. k-linear stable ∞-categories, k field of characteristic 0 (J. Wallbridge 2012, J. Lurie 2011, I. Iwanari 2014).
4. Special cases over ring spectra (J. Wallbridge 2012).
5. Tannakian reconstruction of spectral schemes and stacks (J. Lurie 2004, Bhatt-Halpern-Leistner 2015, Antieau-Stefanich 2025).

Objectives:

- Come back (finally!) to one of the original motivations for DAG and establish a general form of tannakian duality for k-linear stable ∞-categories (k arbitrary base ring).
- Use this to introduce *exponential homotopy theory* (Ludmil's question).

What's tannakian duality?

Adjunction

$$\mathrm{Fib} : \{k - \text{linear symmetric } \otimes - \infty - \text{categories}\} \leftrightarrows \{\text{derived } \infty - \text{stacks over } \mathbf{Spec}\, k\} : \mathcal{D}$$

What's tannakian duality?

Adjunction

$$\mathrm{Fib} : \{k - \text{linear symmetric } \otimes - \infty - \text{categories}\} \leftrightarrows \{\text{derived } \infty - \text{stacks over } \mathbf{Spec}\, k\} : \mathcal{D}$$

Fib = "fiber functors" $\mathcal{D}$ = some categories of sheaves (QCoh, Parf ...).

What's tannakian duality?

Adjunction

$$\mathrm{Fib} : \{k - \text{linear symmetric } \otimes - \infty - \text{categories}\} \leftrightarrows \{\text{derived } \infty - \text{stacks over } \mathbf{Spec}\, k\} : \mathcal{D}$$

Fib = "fiber functors" $\mathcal{D}$ = some categories of sheaves (QCoh, Parf ...).

Two sides:

1. **Tannakian reconstruction:** conditions on a derived stack X so that $X \simeq \mathrm{Fib}(\mathcal{D}(F))$.

What's tannakian duality?

Adjunction

$$\mathrm{Fib} : \{k - \text{linear symmetric } \otimes - \infty - \text{categories}\} \leftrightarrows \{\text{derived } \infty - \text{stacks over } \mathbf{Spec}\, k\} : \mathcal{D}$$

Fib = "fiber functors" $\mathcal{D}$ = some categories of sheaves (QCoh, Parf ...).

Two sides:

1. **Tannakian reconstruction:** conditions on a derived stack X so that $X \simeq \mathrm{Fib}(\mathcal{D}(F))$.
2. **Tannakian deconstruction:** conditions on a $\otimes - \infty$-category $\mathcal{C}$ so that $\mathcal{C} \simeq \mathcal{D}(\mathrm{Fib}(\mathcal{C}))$.

A major obstruction

Well-known fact (J. Lurie): the functor $A \mapsto \mathcal{D}(A)$ induces a fully faithful embedding

$$\{k - \text{linear } \mathbb{E}_\infty - \text{algebras}\} \hookrightarrow \{\text{symmetric} - \otimes\ k - \text{linear presentable } \infty - \text{ categories}\}$$

A major obstruction

Well-known fact (J. Lurie): the functor $A \mapsto \mathcal{D}(A)$ induces a fully faithful embedding

$$\{k - \text{linear } \mathbb{E}_\infty - \text{algebras}\} \hookrightarrow \{\text{symmetric} - \otimes\ k - \text{linear presentable } \infty - \text{ categories}\}$$

However, the forgetful functor

$$\{\text{Derived k-algebras}\} \longrightarrow \{k - \text{linear } \mathbb{E}_\infty - \text{algebras}\}$$

is not fully faithful.

A major obstruction

Well-known fact (J. Lurie): the functor $A \mapsto \mathcal{D}(A)$ induces a fully faithful embedding

$$\{k - \text{linear } \mathbb{E}_\infty - \text{algebras}\} \hookrightarrow \{\text{symmetric} - \otimes\ k - \text{linear presentable } \infty - \text{ categories}\}$$

However, the forgetful functor

$$\{\text{Derived k-algebras}\} \longrightarrow \{k - \text{linear } \mathbb{E}_\infty - \text{algebras}\}$$

is not fully faithful. Left hand side contains {Derived affine schemes} and {affine stacks}: this prevents any reasonable tannakian reconstruction or deconstruction for derived stacks (even for higher underived stacks).

Conceptual explanation

The ∞-category of (quasi-coherent) sheaves $\mathcal{D}(X)$ possesses additional structures that should not be forgotten.

Conceptual explanation

The ∞-category of (quasi-coherent) sheaves $\mathcal{D}(X)$ possesses additional structures that should not be forgotten.

$$\Theta^i := LSym^i : \mathcal{D}(X) \to \mathcal{D}(X)$$

derived symmetric powers ($i \geq 0$). Non-linear ∞-functors that can't be reconstructed from $\otimes$ alone.

Conceptual explanation

The ∞-category of (quasi-coherent) sheaves $\mathcal{D}(X)$ possesses additional structures that should not be forgotten.

$$\Theta^i := LSym^i : \mathcal{D}(X) \to \mathcal{D}(X)$$

derived symmetric powers ($i \geq 0$). Non-linear ∞-functors that can't be reconstructed from $\otimes$ alone.

$$LSym^i \neq Sym^i_{\mathbb{E}_\infty} \qquad (e.g.\ LSym^i(\mathcal{O}_X) = \mathcal{O}_X \quad \forall i \geq 0)$$

A first piece of solution: Θ-categories

The operations Θ^i combine into an ∞-monad $\Theta = \oplus\Theta^i$.

A first piece of solution: Θ-categories

The operations Θ^i combine into an ∞-monad $\Theta = \oplus\Theta^i$. There is a canonical morphism of monads $\mathbb{E}_\infty \to \Theta$ such that

$$\Theta - Alg \to \mathbb{E}_\infty - Alg$$

commutes with arbitrary colimits.

A first piece of solution: Θ-categories

The operations Θ^i combine into an ∞-monad $\Theta = \oplus\Theta^i$. There is a canonical morphism of monads $\mathbb{E}_\infty \to \Theta$ such that

$$\Theta - Alg \to \mathbb{E}_\infty - Alg$$

commutes with arbitrary colimits.

Definition (Nuiten, T.)

Such a pair $(\mathcal{C}, \Theta)$, consisting of

- a symmetric monoidal ∞-category $\mathcal{C}$
- a monad Θ with a map $\mathbb{E}_\infty \to \Theta$

as above is called a **Θ-category**.

Θ-Categories

A first step toward the objective.

Θ-Categories

A first step toward the objective.

Proposition (Nuiten, T.)

$$\{\textit{Derived } k\textit{-algebras}\} \longrightarrow \{k\textit{-linear } \Theta - \textit{categories}\}$$

sending A *to* $(\mathcal{D}(A), LSym)$ *is fully faithful.*

Θ-Categories

A first step toward the objective.

Proposition (Nuiten, T.)

$$\{\textit{Derived } k\textit{-algebras}\} \longrightarrow \{k\textit{-linear } \Theta - \textit{categories}\}$$

sending A to $(\mathcal{D}(A), LSym)$ is fully faithful.

However, tannakian duality is more complicated and Θ-categories only provides incomplete results.

Θ-Categories

A first step toward the objective.

Proposition (Nuiten, T.)

$$\{\textit{Derived } k\textit{-algebras}\} \longrightarrow \{k\textit{-linear } \Theta - \textit{categories}\}$$

sending A to $(\mathcal{D}(A), LSym)$ is fully faithful.

However, tannakian duality is more complicated and Θ-categories only provides incomplete results. Reasons: Θ does not determine the individual pieces Θ^i.

Θ-Categories

A first step toward the objective.

Proposition (Nuiten, T.)

$$\{\textit{Derived } k\textit{-algebras}\} \longrightarrow \{k\textit{-linear } \Theta - \textit{categories}\}$$

sending A to $(\mathcal{D}(A), LSym)$ is fully faithful.

However, tannakian duality is more complicated and Θ-categories only provides incomplete results. Reasons: Θ does not determine the individual pieces Θ^i. Cause problems to say, for instance, that the Θ-structure "preserves the dualizable objects".

A second piece of solution: algebraic ∞-functors

1. Need a setting to ensure that the data of Θ is equivalent to the data of the homogeneous pieces Θ^i.

A second piece of solution: algebraic ∞-functors

1. Need a setting to ensure that the data of Θ is equivalent to the data of the homogeneous pieces Θ^i.
2. Want to use Goodwillie calculus: but it gives the wrong answer.

A second piece of solution: algebraic ∞-functors

1. Need a setting to ensure that the data of Θ is equivalent to the data of the homogeneous pieces Θ^i.
2. Want to use Goodwillie calculus: but it gives the wrong answer.
3. Need a k-linear version of Goodwillie: "strictly polynomial ∞-functors over k".

A second piece of solution: algebraic ∞-functors

1. Need a setting to ensure that the data of Θ is equivalent to the data of the homogeneous pieces Θ^i.
2. Want to use Goodwillie calculus: but it gives the wrong answer.
3. Need a k-linear version of Goodwillie: "strictly polynomial ∞-functors over k".

For two perfect complexes of k-modules M and N, there is a notion of **polynomial map $M \to N$ homogeneous of degree q**

A second piece of solution: algebraic ∞-functors

1. Need a setting to ensure that the data of Θ is equivalent to the data of the homogeneous pieces Θ^i.
2. Want to use Goodwillie calculus: but it gives the wrong answer.
3. Need a k-linear version of Goodwillie: "strictly polynomial ∞-functors over k".

For two perfect complexes of k-modules M and N, there is a notion of **polynomial map $M \to N$ homogeneous of degree q**

$$\mathbb{L}\Gamma^q(M) \to N$$

where $\mathbb{L}\Gamma$ is the derived divided power operation.

A second piece of solution: algebraic ∞-functors

1. Need a setting to ensure that the data of Θ is equivalent to the data of the homogeneous pieces Θ^i.
2. Want to use Goodwillie calculus: but it gives the wrong answer.
3. Need a k-linear version of Goodwillie: "strictly polynomial ∞-functors over k".

For two perfect complexes of k-modules M and N, there is a notion of **polynomial map $M \to N$ homogeneous of degree q**

$$\mathbb{L}\Gamma^q(M) \to N$$

where $\mathbb{L}\Gamma$ is the derived divided power operation. Gives rise to a notion of strictly polynomial ∞-functors between k-linear stable ∞-categories: polynomials on each mapping complexes.

Algebraic ∞-category theory

k-linear stable (compactly generated) ∞-categories and algebraic ∞-functors form a $(2, \infty)$-category.

Algebraic ∞-category theory

k-linear stable (compactly generated) ∞-categories and algebraic ∞-functors form a $(2,\infty)$-category. It is locally $\mathbb{N}$-graded: each morphism f splits canonically

$$f = \oplus f_q$$

with f_q homogeneous of degree q.

Algebraic ∞-category theory

k-linear stable (compactly generated) ∞-categories and algebraic ∞-functors form a $(2,\infty)$-category. It is locally $\mathbb{N}$-graded: each morphism f splits canonically

$$f = \oplus f_q$$

with f_q homogeneous of degree q. Leads to a nice notion of **algebraic ∞-category theory over k**.

Algebraic ∞-category theory

k-linear stable (compactly generated) ∞-categories and algebraic ∞-functors form a $(2,\infty)$-category. It is locally $\mathbb{N}$-graded: each morphism f splits canonically

$$f = \oplus f_q$$

with f_q homogeneous of degree q. Leads to a nice notion of **algebraic ∞-category theory over k**.

Definition (Nuiten, Vezzosi, T.)

An algebraic Θ-category is a symmetric-$\otimes$ k-linear stable ∞-category $\mathcal{C}$ endowed with an algebraic ∞-monad Θ and a map $\mathbb{E}_\infty \to \Theta$ such that $\Theta - Alg(\mathcal{C}) \to \mathbb{E}_\infty - Alg(\mathcal{C})$ preserves all colimits.

Constructing algebraic Θ-categories

Start with (nice enough) $\mathcal{C}_0$ a small symmetric 1-category.

Proposition

The convolution $\otimes$ on $Fun(\mathcal{C}_0^{op}, \mathcal{D}(k))$ admits a canonical Θ-refinement.

Constructing algebraic Θ-categories

Start with (nice enough) $\mathcal{C}_0$ a small symmetric 1-category.

Proposition

The convolution $\otimes$ on $Fun(\mathcal{C}_0^{op}, \mathcal{D}(k))$ admits a canonical Θ-refinement.

More generally: $\exists$ a Θ-convolution structure on $Fun(\mathcal{C}_1, \mathcal{C}_2)$ for two algebraic Θ-categories $\mathcal{C}_1$ and $\mathcal{C}_2$.

Constructing algebraic Θ-categories

Start with (nice enough) $\mathcal{C}_0$ a small symmetric 1-category.

Proposition

The convolution $\otimes$ on $Fun(\mathcal{C}_0^{op}, \mathcal{D}(k))$ admits a canonical Θ-refinement.

More generally: $\exists$ a Θ-convolution structure on $Fun(\mathcal{C}_1, \mathcal{C}_2)$ for two algebraic Θ-categories $\mathcal{C}_1$ and $\mathcal{C}_2$. The Θ-algebras are the "lax Θ-functors".

Tannakian Θ-categories and gerbes

An algebraic Θ-category $\mathcal{C}$ with compatible t-structure is **tannakian** if:

1. $\mathcal{C}$ is rigid (i.e. compact objects are dualizable objects).
2. There is $k \to k'$ faithfully flat and a Θ-functor $\omega : \mathcal{C} \to \mathcal{D}(k')$ such that $\mathcal{C} \otimes_k k' \to \mathcal{D}(k')$ is exact and conservative.

Tannakian Θ-categories and gerbes

An algebraic Θ-category $\mathcal{C}$ with compatible t-structure is **tannakian** if:

1. $\mathcal{C}$ is rigid (i.e. compact objects are dualizable objects).
2. There is $k \to k'$ faithfully flat and a Θ-functor $\omega : \mathcal{C} \to \mathcal{D}(k')$ such that $\mathcal{C} \otimes_k k' \to \mathcal{D}(k')$ is exact and conservative.

A stack $X \in \mathbf{St}_k$ (underived for the *fpqc*-topology) is a **tannakian gerbe** if:

1. $\pi_0(X) \simeq *$.
2. For each $x \in X(k')$ the group stack $aut(x)$ is affine and of positive Tor dimension over k'.
3. X is *Parf*-local.

Tannakian duality

Theorem (Nuiten, Vezzosi, T.)

Let $(\mathcal{C}, \Theta)$ be an algebraic tannakian Θ-category over k.

1. $\mathrm{Fib}(\mathcal{C})$ *is a tannakian gerbe.*
2. *The adjunction*

$$\mathcal{C} \to \mathit{Ind}\mathrm{Perf}(\mathrm{Fib}(\mathcal{C}))$$

is an equivalence of algebraic Θ-categories.

Example 1: schematization $X^{sch}_{\mathbb{Z}}$

X a finite CW complex. Consider the ∞-category $\mathcal{D}^{sm}(X, \mathbb{Z})$ of bounded complexes of sheaves of abelian groups with finite type and locally constant cohomology sheaves.

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

X a finite CW complex. Consider the ∞-category $\mathcal{D}^{sm}(X, \mathbb{Z})$ of bounded complexes of sheaves of abelian groups with finite type and locally constant cohomology sheaves. The Tannakian dual $X_{\mathbb{Z}}^{sch}$ is the **schematization** of X over $\mathbb{Z}$.

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

X a finite CW complex. Consider the ∞-category $\mathcal{D}^{sm}(X, \mathbb{Z})$ of bounded complexes of sheaves of abelian groups with finite type and locally constant cohomology sheaves. The Tannakian dual $X_{\mathbb{Z}}^{sch}$ is the **schematization** of X over $\mathbb{Z}$.

Example: $X = S^1 = B\mathbb{Z}$ then $X_{\mathbb{Z}}^{sch} = B\mathbb{G}$ where $\mathbb{G}$ is an explicit affine commutative flat group scheme over $\mathbb{Z}$: Cartier dual of $\widehat{\mathbb{G}}_m$ the "extended formal completion of $\mathbb{G}_m$".

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

X a finite CW complex. Consider the ∞-category $\mathcal{D}^{sm}(X, \mathbb{Z})$ of bounded complexes of sheaves of abelian groups with finite type and locally constant cohomology sheaves. The Tannakian dual $X_{\mathbb{Z}}^{sch}$ is the **schematization** of X over $\mathbb{Z}$.

Example: $X = S^1 = B\mathbb{Z}$ then $X_{\mathbb{Z}}^{sch} = B\mathbb{G}$ where $\mathbb{G}$ is an explicit affine commutative flat group scheme over $\mathbb{Z}$: Cartier dual of $\widehat{\mathbb{G}}_m$ the "extended formal completion of $\mathbb{G}_m$".

$$\widehat{\mathbb{G}}_m = \underset{Spec\ A \to \mathbb{G}_m}{"colim"}\ Spec\ A$$

where the colimit is taken over finite flat $\mathbb{Z}$-algebras.

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

X a finite CW complex. Consider the ∞-category $\mathcal{D}^{sm}(X, \mathbb{Z})$ of bounded complexes of sheaves of abelian groups with finite type and locally constant cohomology sheaves. The Tannakian dual $X_{\mathbb{Z}}^{sch}$ is the **schematization** of X over $\mathbb{Z}$.

Example: $X = S^1 = B\mathbb{Z}$ then $X_{\mathbb{Z}}^{sch} = B\mathbb{G}$ where $\mathbb{G}$ is an explicit affine commutative flat group scheme over $\mathbb{Z}$: Cartier dual of $\widehat{\mathbb{G}}_m$ the "extended formal completion of $\mathbb{G}_m$".

$$\widehat{\mathbb{G}}_m = \underset{Spec\, A \to \mathbb{G}_m}{"colim"} Spec\, A$$

where the colimit is taken over finite flat $\mathbb{Z}$-algebras.

$$\mathbb{G} = Spec\, \mathcal{O}(\widehat{\mathbb{G}}_m)^{\vee}$$

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

In general $X_{\mathbb{Z}}^{sch}$ is enormous and $\pi_i(X_{\mathbb{Z}}^{sch})$ not easy (impossible) to describe for $i > 1$.

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

In general $X_{\mathbb{Z}}^{sch}$ is enormous and $\pi_i(X_{\mathbb{Z}}^{sch})$ not easy (impossible) to describe for $i > 1$.
When $X_{\mathbb{Z}}^{sch}$ has a "maximal unipotent quotient" called the affinization

$$X_{\mathbb{Z}}^{aff} \simeq Spec\ C^*(X, \mathbb{Z}).$$

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

In general $X_{\mathbb{Z}}^{sch}$ is enormous and $\pi_i(X_{\mathbb{Z}}^{sch})$ not easy (impossible) to describe for $i > 1$.
When $X_{\mathbb{Z}}^{sch}$ has a "maximal unipotent quotient" called the affinization

$$X_{\mathbb{Z}}^{aff} \simeq Spec\, C^*(X, \mathbb{Z}).$$

When X is 1-connected (or nilpotent) we have

$$\pi_i(X_{\mathbb{Z}}^{aff}) \simeq \pi_i(X) \otimes \mathbb{H}$$

where $\mathbb{H} = Spec\, \mathbb{Z} < x >$ the spectrum of the free binomial ring over one generator.

Example 1: schematization $X_{\mathbb{Z}}^{sch}$

In general $X_{\mathbb{Z}}^{sch}$ is enormous and $\pi_i(X_{\mathbb{Z}}^{sch})$ not easy (impossible) to describe for $i > 1$.
When $X_{\mathbb{Z}}^{sch}$ has a "maximal unipotent quotient" called the affinization

$$X_{\mathbb{Z}}^{aff} \simeq Spec\, C^*(X, \mathbb{Z}).$$

When X is 1-connected (or nilpotent) we have

$$\pi_i(X_{\mathbb{Z}}^{aff}) \simeq \pi_i(X) \otimes \mathbb{H}$$

where $\mathbb{H} = Spec\, \mathbb{Z} < x >$ the spectrum of the free binomial ring over one generator.
$X \mapsto X_{\mathbb{Z}}^{aff}$ is close to being a full embedding (e.g. injective on equivalence classes of objects) and can be refined to a full embedding by fixing trivializations of Frobenius.

Exponential sheaves and complexes

Idea of "exponential motives": Kontsevich, Fresán-Jossen, Gallauer-Pepin Lehalleur ....

Exponential sheaves and complexes

Idea of "exponential motives": Kontsevich, Fresán-Jossen, Gallauer-Pepin Lehalleur ....
Work over $\mathbb{C}$ (similar constructions can be made in different contexts, e.g. ℓ-adic, mixed Hodge, motivic).

Exponential sheaves and complexes

Idea of "exponential motives": Kontsevich, Fresán-Jossen, Gallauer-Pepin Lehalleur ....
Work over $\mathbb{C}$ (similar constructions can be made in different contexts, e.g. ℓ-adic, mixed Hodge, motivic).

The ∞-category $\mathcal{D}_{exp}$ is defined to be the stable quotient $\mathcal{D}_c(\mathbb{A}^1)/\mathcal{D}_c(*)$ where $\mathcal{D}_c(X)$ is the ∞-category of (algebraically) constructible complexes of sheaves of $\mathbb{Z}$-modules on the variety X.

Exponential sheaves and complexes

Idea of "exponential motives": Kontsevich, Fresán-Jossen, Gallauer-Pepin Lehalleur .... Work over $\mathbb{C}$ (similar constructions can be made in different contexts, e.g. ℓ-adic, mixed Hodge, motivic).

The ∞-category $\mathcal{D}_{exp}$ is defined to be the stable quotient $\mathcal{D}_c(\mathbb{A}^1)/\mathcal{D}_c(*)$ where $\mathcal{D}_c(X)$ is the ∞-category of (algebraically) constructible complexes of sheaves of $\mathbb{Z}$-modules on the variety X.

Concrete realizations: the projection $\mathcal{D}_c(\mathbb{A}^1) \to \mathcal{D}_{exp}$ possesses a left and a right adjoint which are fully faithful

$$\mathcal{D}_{exp} \simeq \{E \in \mathcal{D}_c(\mathbb{A}^1)/H^*(\mathbb{A}^1, E) \simeq 0\} \qquad \mathcal{D}_{exp} \simeq \{E \in \mathcal{D}_c(\mathbb{A}^1)/H^*_c(\mathbb{A}^1, E) \simeq 0\}$$

Exponential sheaves and complexes

Idea of "exponential motives": Kontsevich, Fresán-Jossen, Gallauer-Pepin Lehalleur .... Work over $\mathbb{C}$ (similar constructions can be made in different contexts, e.g. ℓ-adic, mixed Hodge, motivic).

The ∞-category $\mathcal{D}_{exp}$ is defined to be the stable quotient $\mathcal{D}_c(\mathbb{A}^1)/\mathcal{D}_c(*)$ where $\mathcal{D}_c(X)$ is the ∞-category of (algebraically) constructible complexes of sheaves of $\mathbb{Z}$-modules on the variety X.

Concrete realizations: the projection $\mathcal{D}_c(\mathbb{A}^1) \to \mathcal{D}_{exp}$ possesses a left and a right adjoint which are fully faithful

$$\mathcal{D}_{exp} \simeq \{E \in \mathcal{D}_c(\mathbb{A}^1)/H^*(\mathbb{A}^1, E) \simeq 0\} \qquad \mathcal{D}_{exp} \simeq \{E \in \mathcal{D}_c(\mathbb{A}^1)/H_c^*(\mathbb{A}^1, E) \simeq 0\}$$

The additive convolution tensor product $\star$ on $\mathcal{D}_c(\mathbb{A}^1)$ descends to a symmetric monoidal structure on $\mathcal{D}_{exp}$.

Exponential sheaves and complexes

Idea of "exponential motives": Kontsevich, Fresán-Jossen, Gallauer-Pepin Lehalleur .... Work over $\mathbb{C}$ (similar constructions can be made in different contexts, e.g. ℓ-adic, mixed Hodge, motivic).

The ∞-category $\mathcal{D}_{exp}$ is defined to be the stable quotient $\mathcal{D}_c(\mathbb{A}^1)/\mathcal{D}_c(*)$ where $\mathcal{D}_c(X)$ is the ∞-category of (algebraically) constructible complexes of sheaves of $\mathbb{Z}$-modules on the variety X.

Concrete realizations: the projection $\mathcal{D}_c(\mathbb{A}^1) \to \mathcal{D}_{exp}$ possesses a left and a right adjoint which are fully faithful

$$\mathcal{D}_{exp} \simeq \{E \in \mathcal{D}_c(\mathbb{A}^1)/H^*(\mathbb{A}^1, E) \simeq 0\} \qquad \mathcal{D}_{exp} \simeq \{E \in \mathcal{D}_c(\mathbb{A}^1)/H_c^*(\mathbb{A}^1, E) \simeq 0\}$$

The additive convolution tensor product $\star$ on $\mathcal{D}_c(\mathbb{A}^1)$ descends to a symmetric monoidal structure on $\mathcal{D}_{exp}$.

Exponential complexes on X

For an algebraic variety X we define

$$\mathcal{D}_{exp}(X) := \mathcal{D}_c(X \times \mathbb{A}^1)/\pi^*(\mathcal{D}_c(X))$$

where $\pi : X \times \mathbb{A}^1 \to X$.

Exponential complexes on X

For an algebraic variety X we define

$$\mathcal{D}_{exp}(X) := \mathcal{D}_c(X \times \mathbb{A}^1)/\pi^*(\mathcal{D}_c(X))$$

where $\pi : X \times \mathbb{A}^1 \to X$. Convolution tensor product

$$E \star F := \mu_!(E \boxtimes_{\mathcal{O}_X} F)$$

$\mu : \mathbb{A}^1_X \times_X \mathbb{A}^1_X \to \mathbb{A}^1_X$ the additive law.

Exponential complexes on X

For an algebraic variety X we define

$$\mathcal{D}_{exp}(X) := \mathcal{D}_c(X \times \mathbb{A}^1)/\pi^*(\mathcal{D}_c(X))$$

where $\pi : X \times \mathbb{A}^1 \to X$. Convolution tensor product

$$E \star F := \mu_!(E \boxtimes_{\mathcal{O}_X} F)$$

$\mu : \mathbb{A}^1_X \times_X \mathbb{A}^1_X \to \mathbb{A}^1_X$ the additive law.

An object $E \in \mathcal{D}_{exp}(X)$ can be thought as a constructible family of objects of $\mathcal{D}_{exp}$ parametrized by X: $\{E_x\}_{x \in X}$. We want to restrict to "smooth families" to get a Tannakian-like category: notion of "exponential local system" on X.

Smooth exponential complexes on X

Definition

$E \in \mathcal{D}_{exp}(X)$ is **smooth** if there is a closed $D \subset X \times \mathbb{A}^1$ which is finite flat over X such that E has locally constant cohomology sheaves outside of D.

Smooth exponential complexes on X

Definition

$E \in \mathcal{D}_{exp}(X)$ is **smooth** if there is a closed $D \subset X \times \mathbb{A}^1$ which is finite flat over X such that E has locally constant cohomology sheaves outside of D.

Idea: the family $\{E_x\}_{x \in X}$ becomes locally constant after applying a fiber functor $\mathcal{D}_{exp} \to \mathcal{D}(*)$.

Smooth exponential complexes on X

Definition

$E \in \mathcal{D}_{exp}(X)$ is **smooth** if there is a closed $D \subset X \times \mathbb{A}^1$ which is finite flat over X such that E has locally constant cohomology sheaves outside of D.

Idea: the family $\{E_x\}_{x \in X}$ becomes locally constant after applying a fiber functor $\mathcal{D}_{exp} \to \mathcal{D}(*)$.
Warning: It is not a locally constant family of objects in $\mathcal{D}_{exp}$.

Smooth exponential complexes on X

Definition

$E \in \mathcal{D}_{exp}(X)$ is **smooth** if there is a closed $D \subset X \times \mathbb{A}^1$ which is finite flat over X such that E has locally constant cohomology sheaves outside of D.

Idea: the family $\{E_x\}_{x \in X}$ becomes locally constant after applying a fiber functor $\mathcal{D}_{exp} \to \mathcal{D}(*)$.
Warning: It is not a locally constant family of objects in $\mathcal{D}_{exp}$.

A smooth exponential complex E possesses an underlying "local system" on X, namely the restriction of E near $X \times \{\infty\}$.

Smooth exponential complexes on X

Definition

$E \in \mathcal{D}_{exp}(X)$ is **smooth** if there is a closed $D \subset X \times \mathbb{A}^1$ which is finite flat over X such that E has locally constant cohomology sheaves outside of D.

Idea: the family $\{E_x\}_{x \in X}$ becomes locally constant after applying a fiber functor $\mathcal{D}_{exp} \to \mathcal{D}(*)$.
Warning: It is not a locally constant family of objects in $\mathcal{D}_{exp}$.

A smooth exponential complex E possesses an underlying "local system" on X, namely the restriction of E near $X \times \{\infty\}$.

$\mathcal{D}^{sm}_{exp}(X) \subset \mathcal{D}_{exp}(X)$ the full sub ∞-category of smooth objects. It is a rigid symmetric monoidal stable $\mathbb{Z}$-linear ∞-category.

Smooth exponential complexes on X

Definition

$E \in \mathcal{D}_{exp}(X)$ is **smooth** if there is a closed $D \subset X \times \mathbb{A}^1$ which is finite flat over X such that E has locally constant cohomology sheaves outside of D.

Idea: the family $\{E_x\}_{x \in X}$ becomes locally constant after applying a fiber functor $\mathcal{D}_{exp} \to \mathcal{D}(*)$.
Warning: It is not a locally constant family of objects in $\mathcal{D}_{exp}$.

A smooth exponential complex E possesses an underlying "local system" on X, namely the restriction of E near $X \times \{\infty\}$.

$\mathcal{D}^{sm}_{exp}(X) \subset \mathcal{D}_{exp}(X)$ the full sub ∞-category of smooth objects. It is a rigid symmetric monoidal stable $\mathbb{Z}$-linear ∞-category. A fiber functor is given by

$$\omega_{x,\infty} : \mathcal{D}^{sm}_{exp}(X) \to \mathcal{D}_{exp}(\{x\}) \to \mathcal{D}_c(*)$$

"fiber at (x, ∞)".

Smooth exponential complexes on X

Typical construction of smooth objects.

1. L a local system on $X \times \mathbb{G}_m$, and $j_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$.

Smooth exponential complexes on X

Typical construction of smooth objects.

1. L a local system on $X \times \mathbb{G}_m$, and $j_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$.
2. $i : D \subset X \times \mathbb{A}^1$ finite flat over X and $L \in \mathcal{D}_c(D)$ such that $\pi_*(L)$ is a local system on X. $i_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$.

Smooth exponential complexes on X

Typical construction of smooth objects.

1. L a local system on $X \times \mathbb{G}_m$, and $j_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$.
2. $i : D \subset X \times \mathbb{A}^1$ finite flat over X and $L \in \mathcal{D}_c(D)$ such that $\pi_*(L)$ is a local system on X. $i_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$.
3. $D \subset X \times \mathbb{A}^1$ finite and étale over X and L a local system on $X \times \mathbb{A}^1 - D$. $j_!(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{A}^1 - D \hookrightarrow X \times \mathbb{A}^1$.

Smooth exponential complexes on X

Typical construction of smooth objects.

1. L a local system on $X \times \mathbb{G}_m$, and $j_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$.
2. $i : D \subset X \times \mathbb{A}^1$ finite flat over X and $L \in \mathcal{D}_c(D)$ such that $\pi_*(L)$ is a local system on X. $i_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$.
3. $D \subset X \times \mathbb{A}^1$ finite and étale over X and L a local system on $X \times \mathbb{A}^1 - D$. $j_!(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{A}^1 - D \hookrightarrow X \times \mathbb{A}^1$.

Two comments:

- When $D \subset X \times \mathbb{A}^1$ is the graph of $f : X \to \mathbb{A}^1$, $i_*(\mathbb{Z}_X)$ is denoted by $\mathcal{E}^f$ ("$exp(f)$"). $\mathcal{E}^f \star \mathcal{E}^g \simeq \mathcal{E}^{f+g}$.

Smooth exponential complexes on X

Typical construction of smooth objects.

1. L a local system on $X \times \mathbb{G}_m$, and $j_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$.
2. $i : D \subset X \times \mathbb{A}^1$ finite flat over X and $L \in \mathcal{D}_c(D)$ such that $\pi_*(L)$ is a local system on X. $i_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$.
3. $D \subset X \times \mathbb{A}^1$ finite and étale over X and L a local system on $X \times \mathbb{A}^1 - D$. $j_!(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{A}^1 - D \hookrightarrow X \times \mathbb{A}^1$.

Two comments:

- When $D \subset X \times \mathbb{A}^1$ is the graph of $f : X \to \mathbb{A}^1$, $i_*(\mathbb{Z}_X)$ is denoted by $\mathcal{E}^f$ ("$exp(f)$"). $\mathcal{E}^f \star \mathcal{E}^g \simeq \mathcal{E}^{f+g}$.
- $Loc(X) \subset \mathcal{D}^{sm}_{exp}(X)$ (via $D = X$). Usual local systems on X are exponential local systems.

Smooth exponential complexes on X

Typical construction of smooth objects.

1. L a local system on $X \times \mathbb{G}_m$, and $j_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{G}_m \hookrightarrow X \times \mathbb{A}^1$.
2. $i : D \subset X \times \mathbb{A}^1$ finite flat over X and $L \in \mathcal{D}_c(D)$ such that $\pi_*(L)$ is a local system on X. $i_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$.
3. $D \subset X \times \mathbb{A}^1$ finite and étale over X and L a local system on $X \times \mathbb{A}^1 - D$. $j_!(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$ where $j : X \times \mathbb{A}^1 - D \hookrightarrow X \times \mathbb{A}^1$.

Two comments:

- When $D \subset X \times \mathbb{A}^1$ is the graph of $f : X \to \mathbb{A}^1$, $i_*(\mathbb{Z}_X)$ is denoted by $\mathcal{E}^f$ ("$exp(f)$"). $\mathcal{E}^f \star \mathcal{E}^g \simeq \mathcal{E}^{f+g}$.
- $Loc(X) \subset \mathcal{D}^{sm}_{exp}(X)$ (via $D = X$). Usual local systems on X are exponential local systems.

In general, objects in $\mathcal{D}^{sm}_{exp}(X)$ look like local systems on X "twisted" by functions on X (possibly multivalued).

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

$\mathcal{D}_{exp}^{sm}(X, \mathbb{Z})$ has a canonical Θ-structure and is tannakian over $\mathbb{Z}$. By tannakian duality we get a tannakian gerbe $X_{\mathbb{Z}}^{exp}$.

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

$\mathcal{D}_{exp}^{sm}(X, \mathbb{Z})$ has a canonical Θ-structure and is tannakian over $\mathbb{Z}$. By tannakian duality we get a tannakian gerbe $X_{\mathbb{Z}}^{exp}$. The natural functor $\mathcal{D}_{exp} \to \mathcal{D}_{exp}^{sm}(X)$ provides a canonical morphism

$$X_{\mathbb{Z}}^{exp} \to *_{\mathbb{Z}}^{exp}.$$

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

$\mathcal{D}_{exp}^{sm}(X,\mathbb{Z})$ has a canonical Θ-structure and is tannakian over $\mathbb{Z}$. By tannakian duality we get a tannakian gerbe $X_{\mathbb{Z}}^{exp}$. The natural functor $\mathcal{D}_{exp} \to \mathcal{D}_{exp}^{sm}(X)$ provides a canonical morphism

$$X_{\mathbb{Z}}^{exp} \to *_{\mathbb{Z}}^{exp}.$$

I suspect $*_{\mathbb{Z}}^{exp} = B\mathbb{G}_{exp,\mathbb{Z}}$ for $\mathbb{G}_{exp,\mathbb{Z}}$ a flat affine group scheme over $\mathbb{Z}$. It contains a copy of $\mathbb{G}$ ("global monodromy") and a torus (over $\mathbb{Z}$) with character group $(\mathbb{C},+)$.

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

$\mathcal{D}_{exp}^{sm}(X,\mathbb{Z})$ has a canonical Θ-structure and is tannakian over $\mathbb{Z}$. By tannakian duality we get a tannakian gerbe $X_{\mathbb{Z}}^{exp}$. The natural functor $\mathcal{D}_{exp} \to \mathcal{D}_{exp}^{sm}(X)$ provides a canonical morphism

$$X_{\mathbb{Z}}^{exp} \to *_{\mathbb{Z}}^{exp}.$$

I suspect $*_{\mathbb{Z}}^{exp} = B\mathbb{G}_{exp,\mathbb{Z}}$ for $\mathbb{G}_{exp,\mathbb{Z}}$ a flat affine group scheme over $\mathbb{Z}$. It contains a copy of $\mathbb{G}$ ("global monodromy") and a torus (over $\mathbb{Z}$) with character group $(\mathbb{C},+)$.

Define the geometric version by a pull-back square

$$\begin{array}{ccc} X_{\mathbb{Z}}^{exp} & \longrightarrow & *_{\mathbb{Z}}^{exp} \\ \uparrow & & \uparrow \\ \bar{X}_{\mathbb{Z}}^{exp} & \longrightarrow & * \end{array}$$

$\mathbb{G}_{exp,\mathbb{Z}}$ acts on $\bar{X}_{\mathbb{Z}}^{exp}$.

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

For a local system L on X and a function $f : X \to \mathbb{A}^1$ we can define the object

$$\varepsilon^f(L) = (j_f)_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$$

where $j_f : X \hookrightarrow X \times \mathbb{A}^1$ is the graph of f.

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

For a local system L on X and a function $f : X \to \mathbb{A}^1$ we can define the object

$$\varepsilon^f(L) = (j_f)_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$$

where $j_f : X \hookrightarrow X \times \mathbb{A}^1$ is the graph of f. This provides a natural projection

$$X_{\mathbb{Z}}^{exp} \to X_{\mathbb{Z}}^{sch} \times BT_X,$$

where $T_X = Spec\, \mathbb{Z}[\mathcal{O}(X)]$ is the torus with character group $(\mathcal{O}(X), +)$.

Example 2: exponential homotopy type $X_{\mathbb{Z}}^{exp}$

For a local system L on X and a function $f : X \to \mathbb{A}^1$ we can define the object

$$\varepsilon^f(L) = (j_f)_*(L) \in \mathcal{D}_c(X \times \mathbb{A}^1)$$

where $j_f : X \hookrightarrow X \times \mathbb{A}^1$ is the graph of f. This provides a natural projection

$$X_{\mathbb{Z}}^{exp} \to X_{\mathbb{Z}}^{sch} \times BT_X,$$

where $T_X = Spec\, \mathbb{Z}[\mathcal{O}(X)]$ is the torus with character group $(\mathcal{O}(X), +)$. The fiber at $X \times \infty$ provides a functor $\mathcal{D}_{exp}^{sm}(X) \to \mathcal{D}^{sm}(X)$ and thus a morphism $X_{\mathbb{Z}}^{sch} \to X_{\mathbb{Z}}^{exp}$ making $X_{\mathbb{Z}}^{sch}$ into a retract of $X_{\mathbb{Z}}^{exp}$.

Further developments

1. $\mathbb{G}_{exp}$ as the automorphism of $X \mapsto \pi_1(\bar{X}_{\mathbb{Z}}^{exp})$ (exponential version of "anabelian geometry").

Further developments

1. $\mathbb{G}_{exp}$ as the automorphism of $X \mapsto \pi_1(\bar{X}_{\mathbb{Z}}^{exp})$ (exponential version of "anabelian geometry").
2. Non-abelian AJ maps (filtrations on the set of rational points):

$$X(k) \mapsto \pi_0((\bar{X}^{exp})^{h\mathbb{G}_{exp}})$$
$$E_2^{p,q} = H^p(\mathbb{G}_{exp}, \pi_{-q}(\bar{X}^{exp})) \Rightarrow X(k)$$

(T., Pal in the étale homotopy context).

Further developments

1. $\mathbb{G}_{exp}$ as the automorphism of $X \mapsto \pi_1(\bar{X}_{\mathbb{Z}}^{exp})$ (exponential version of "anabelian geometry").
2. Non-abelian AJ maps (filtrations on the set of rational points):

$$X(k) \mapsto \pi_0((\bar{X}^{exp})^{h\mathbb{G}_{exp}})$$
$$E_2^{p,q} = H^p(\mathbb{G}_{exp}, \pi_{-q}(\bar{X}^{exp})) \Rightarrow X(k)$$

   (T., Pal in the étale homotopy context).
3. Moduli spaces of objects in $\mathcal{D}_{exp}^{sm}$?

Further developments

1. $\mathbb{G}_{exp}$ as the automorphism of $X \mapsto \pi_1(\bar{X}_{\mathbb{Z}}^{exp})$ (exponential version of "anabelian geometry").
2. Non-abelian AJ maps (filtrations on the set of rational points):

$$X(k) \mapsto \pi_0((\bar{X}^{exp})^{h\mathbb{G}_{exp}})$$
$$E_2^{p,q} = H^p(\mathbb{G}_{exp}, \pi_{-q}(\bar{X}^{exp})) \Rightarrow X(k)$$

   (T., Pal in the étale homotopy context).
3. Moduli spaces of objects in $\mathcal{D}_{exp}^{sm}$? (not so easy, naive moduli functors are not representable).

Further developments

1. $\mathbb{G}_{exp}$ as the automorphism of $X \mapsto \pi_1(\bar{X}_{\mathbb{Z}}^{exp})$ (exponential version of "anabelian geometry").
2. Non-abelian AJ maps (filtrations on the set of rational points):

$$X(k) \mapsto \pi_0((\bar{X}^{exp})^{h\mathbb{G}_{exp}})$$
$$E_2^{p,q} = H^p(\mathbb{G}_{exp}, \pi_{-q}(\bar{X}^{exp})) \Rightarrow X(k)$$

   (T., Pal in the étale homotopy context).
3. Moduli spaces of objects in $\mathcal{D}_{exp}^{sm}$? (not so easy, naive moduli functors are not representable).
4. Motivic version using Nori's approach (π_1 already studied by Fresán-Jossen).

Algebraic ∞-categories, Θ-categories and tannakian duality

Bertrand Toën
joint with J. Nuiten, G. Vezzosi and ChatPPP[2]

CNRS, Université de Toulouse

CATS8, Oléron, September 2026

[2]Old and forgotten model, not quite an LLM but works the same way. It is way cheaper and slow, and makes a lot of mistakes. "PPP" stands for "Piece of Paper and Pen".