Characteristic classes of foliations in positive characteristics

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On the crossroads of algebra, geometry, and physics: A conference dedicated to Alexander Goncharov
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(Joint work with Gabriele Vezzosi ... book in progress)

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For X variety over a perfect field k, want results in $H_{crys}(X)$.

Characteristic classes of holomorphic foliations

Let M be a holomorphic manifold with an n-dimensional holomorphic foliation $F \subset T_X$ given by a subbundle stable by the Lie bracket. Let V a holomorphic vector bundle on X endowed with a flat connection along the leaves: $\nabla : V \to V \otimes F^\vee$ ($+$ Liebniz and $\nabla^2 = 0$).

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Theorem (Bott)

For any homogeneous polynomial ϕ in chern classes of total degree $d > codim(F) = dimX - rank(F)$, we have

$$\phi(c_1(V), \ldots, c_n(V)) = 0 \in H^{2d}_{DR}(X).$$

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Possible extensions to the case where F has singularities (Baum-Bott residues).

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- $X' \to X$ is the Gl_n-torsor associated to $\mathcal{N}_F = T_X/F$ the normal bundle of F.

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When $\mathcal{N}_F$ is trivial we get this way characteristic classes of F in $H^*_{DR}(X)$ (e.g. Godbillon-Vey).

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Objective: Need spell out extra conditions on F insuring that these theorems/constructions remain true in positive characteristic.

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Idea: Consider $X \to Spec\, k$ smooth variety over a perfect field k. $W(k)$ the ring of Witt vectors over k and $X \to Spec\, k \hookrightarrow Spf\, W(k) = S$. We want that the foliation on X "extends over S".

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Notion of **derived foliations**.

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The graded circle

We let $\mathbb{Z} < x >$ be the ring of divided powers over one generator. It is the subring of $\mathbb{Q}[x]$ generated by the elements $x^{(n)} := \frac{x^n}{n!}$ $n \geq 0$.

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where $Ker_p \subset \mathbb{W}$ is the non-reduced subgroup scheme of Witt vectors killed by the Frobenius (successive extension α_p).

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The graded circle S^1_{gr} is not far from a usual circle, but not quite. It has the same cohomology

$$H^*(S^1_{gr}, \mathcal{O}) \simeq H^*(S^1, \mathbb{Z}) \simeq \mathbb{Z} \oplus \mathbb{Z}[-1].$$

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Important comment: This proposition is wrong for the usual topological circle $S^1 = B\mathbb{Z}$ (there is then a non-monoidal equivalence). S^1 and S^1_{gr} are related by a non-trivial filtration ("HKR-filtration" Moulinos-Robalo-T.).

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S^1_{gr} possesses a canonical action of $\mathbb{G}_m$ (given by the natural grading on divided powers). Let $\mathcal{H}$ be the semi-direct product

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The group $\mathcal{H}$ acts on S^1_{gr}. A representation of $\mathcal{H}$ is a "graded mixed complexes" (a mixed complexes with a compatible grading).

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It is equipped with its natural $\mathcal{H}$-action.

The graded loop space and de Rham theory

The underlying derived scheme of $\mathcal{L}^{gr}(X)$ is the shifted tangent space

$$TX[-1] := Spec\, Sym_{\mathcal{O}_X}(\mathbb{L}_X[1]),$$

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For a morphism of schemes $X \to S$, there is a relative graded loop space

$$\mathcal{L}^{gr}(X/S) := \mathcal{L}^{gr}(X) \times_{\mathcal{L}^{gr}(S)} S \simeq T(X/S)[-1].$$

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In particular, when $S = Spec\,\mathbb{C}$ and X of finite type, $\mathbb{H}(\mathcal{L}^{gr}(X/S), \mathcal{O})^{\mathcal{H}}$ computes algebraic de Rham cohomology of X.

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for some quasi-coherent complex $\mathbb{L}_F$ called the cotangent complex of $\mathcal{F}$.

$\mathcal{F}$ should be considered as the data of the complex $\mathbb{L}_{\mathcal{F}}$, together with some complicated extra structure describing a de Rham style differential on $Sym_{\mathcal{O}_X}(\mathbb{L}_{\mathcal{F}}[1])$.

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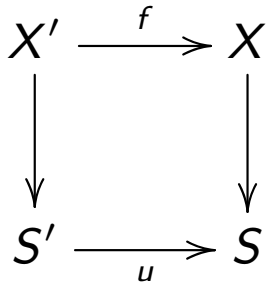

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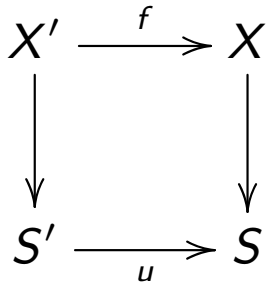

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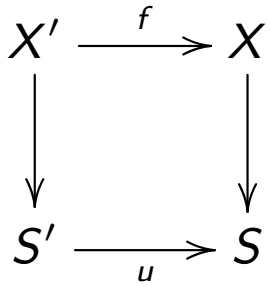

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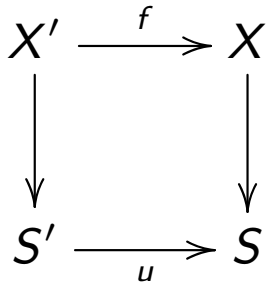

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5 When $X \to S$ is smooth, subbundles which are differential ideals in $\Omega^1_{X/S}$ can be naturally promoted into derived foliations on X relative to S (not as easy statement as it seems).

Foliated cohomology

A derived foliation $\mathcal{F} \in \mathcal{F}ol(X/S)$ has a derived de Rham cohomology complex

$$\widehat{C}^*_{DR,\mathcal{F}}(X) := \mathbb{H}(\mathcal{F}, \mathcal{O})^{h\mathcal{H}}$$

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Warning: completion along the Hodge filtration (à la Illusie).

Hodge filtrations

Proposition

Any $\mathcal{F} \in \mathcal{F}ol(X/S)$ determines a natural descending filtration on $\widehat{C}^*_{DR}(X)$ with

$$Gr^0(\widehat{C}^*_{DR}(X)) \simeq \widehat{C}^*_{DR,\mathcal{F}}(X).$$

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The above proposition can be refined into a more precise statement: the chern classes of E, as cocycles in the complex $\hat{C}^*_{DR,\mathcal{F}}(X/S)$, possess canonical homotopies to 0.

Applications I: Baum-Bott residues

Let k be perfect field, $W(k)$ its ring of Witt vectors and $S = SpfW(k)$. Let X be a smooth algebraic variety over k and $D \subset \Omega^1_{X/k}$ a coherent subsheaf which is a differential ideal. We assume that D is a subbundle over a dense open $U \in X$ with closed complement $Z = X - U$. We let $q = rank(D)$ be the rank of D on U.

Applications I: Baum-Bott residues

Theorem

Assume that the foliation $\mathcal{F} \in \mathcal{F}ol(U/k)$ defined by D extends to a derived foliation $\mathcal{F}' \in \mathcal{F}ol(U/S)$. For any homogeneous polynomial ϕ in Chern classes of degree $d > q$, there exists a well defined class

$$Res_\phi(D) \in H^{2d}_{Z,crys}(X/W(k))$$

lifting $\phi(c_1(D), \ldots, c_q(D)) \in H^{2d}_{crys}(X/W(k))$.

Applications II: characteristic classes

Let X and k as before. Let $K = Frac(W(k))$.

Theorem

Let $\mathcal{F}$ be a smooth foliation defined by a differential ideal $D \subset \Omega^1_{X/k}$, which admits a lift $\mathcal{F}' \in \mathcal{F}ol(X/S)$. Let $X' \to X$ be the Gl_n-torsor associated to the normal bundle of $\mathcal{F}$. Then, there exists a natural algebra map

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- The vanishing theorem for Chern classes in longitudinal de Rham cohomology and the multiplicative property of the filtration.
- The close relations between derived de Rham cohomology of X over S and crystalline cohomology of X relative to S: canonical isomorphism (Illusie-Bhatt)

$$\widehat{H}^*_{dR}(X/S)\widehat{\otimes}_{\widehat{H}^*_{dR}(s/S)}W(k) \simeq H^*_{crys}(X/W(k)).$$

Characteristic classes of foliations in positive characteristics

Bertrand TOEN

On the crossroads of algebra, geometry, and physics: A conference dedicated to Alexander Goncharov
May 23rd 2022