Solutions "up to homotopy" of polynomial equations

(An introduction to Derived Algebraic Geometry)

B. Toën (CNRS, Université de Montpellier 2)

Toulouse, October 2014

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Further references

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- *Derived Algebraic Geometry.* EMS Surveys in Math. Sci. (Preprint arXiv:1401.1044).

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- G. Vezzosi, *What is a derived scheme?* Notices Amer. Math. Soc. 58 (2011), no. 7, 955-958.

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- **Gromov-Witten theory:** virtual fundamental classes (M. Kontsevich, 1994).

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- **Homotopy theory II:** stable homotopy theory, "brave new algebra" (J. Morava, 1970, W. Waldhausen, M. Hopkins, 1988).
- **Mathematical physics:** super geometry, Q-manifolds, "BV" formalism, "AKSZ" formalism (Kostant '70, Schwartz '80, Alexandrov-Kontsevich-Schwartz-Zaboronsky, 1997).

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3. Notion of "derived scheme" and some recent interactions.

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Problem: understand the intersection $Y \cap Z$.

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Typical example: $X = k^2$, $Y_t = \{y - t = 0\}$, $Z_t = \{x^2 - y = 0\}$, $t \in k = S$ and $t_0 = 0$.

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There exist many possible approaches to this "something else".

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- Depend on the choice of $Y_\epsilon \subset X$.
- Does not work well in the algebraic setting (Y can be rigid in X, no non-trivial algebraic perturbations).

Something else: schemes

$$X = k^n \qquad Y = \{F_1 = \cdots = F_p = 0\} \quad Z = \{G_1 = \cdots = G_q = 0\}$$

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Note: $k[X_1, \ldots, X_n]/(F_i, G_j)$ can be non-reduced: $F = X^2 - Y$, $G = Y$, $k[\epsilon]$, $\epsilon^2 = 0$.

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Key notion: solution **up to homotopy** of algebraic equations.

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Local models for schemes (loc. finite type over k).

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- elements $a_i \in A$,
- continuous paths $\alpha_j : [0,1] \longrightarrow A$, $j = 1, \ldots, p$, with

$$\alpha_j(0) = F_j(a_1, \ldots, a_n) \qquad \alpha_j(1) = 0 \qquad \forall j$$

Solutions up to homotopy III

Proposition

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$$\pi_0(\mathcal{A}) \simeq k[X_1, \ldots, X_n]/(F_1, \ldots, F_p).$$

$$\pi_i(\mathcal{A}) \simeq Tor^i_{k[X]}(k[X]/(F_j)) \ \forall i.$$

Solution up to homotopy and derived intersection I

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$$Y \cap^{der} Z := \mathbf{Spec}\, \mathcal{A}$$

where $\mathcal{A}$ is the universal solution up to homotopy for $F_1, \ldots, F_p, G_1, \ldots, G_q$.

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$Y \cap^{der} Z$ is an *affine derived scheme*.

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   implies that $Y \cap^{der} Z$ remembers intersection multiplicities (Serre's Tor formula).
2. $Y \cap^{der} Z$ is always of the "correct" dimension: $dimX - p - q$.
3. $Y \cap^{der} Z$ varies well in algebraic families (no more pathologies at special values of the parameter).

Founding principle

Derived algebraic geometry: scheme theory where commutative rings are replaced by commutative topological rings, **taken up to homotopy**

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Combinatorial model: simplicial commutative rings.
Remark: the homotopy theory of topological rings coincides with that of simplicial rings.

Definition I

Definition (Lying)

A **derived scheme** *consists of a pair* $(X, \mathcal{O}_X)$*, where* X *is a topological space and* $\mathcal{O}_X$ *is a sheaf of commutative topological* k*-algebras on* X*, locally equivalent to* $\operatorname{Spec} \mathcal{A}$*, for* $\mathcal{A}$ *the universal solution up to homotopy of a system of algebraic equations.*

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Remark: "Lying": one also allows the homotopies α_j to satisfy algebraic equations up to homotopy, and so on.

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- The coherent sheaves $\pi_i(\mathcal{O}_X)$ on $t_0(X)$ reflect the *derived structure* (*virtual structure sheaf*, generalized nilpotents).

An important example

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If Z is no longer affine, this is true locally but not globally (Atiyah class).

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A richness of the subject: two morphisms between derived schemes can be homotopic.

Thank you.