20/11/16. B. Toën, Hall algebras in the derived context

Recall: To any finitary ab. cat. $\mathcal{A}$ is possible to associate a Hall alg. $H(\mathcal{A})$

Want: $\mathcal{T}$ a triang. cat. (e.g. $\mathcal{T} = \mathcal{D}^b(\mathrm{mod}\, B)$ where B fd alg), $H(\mathcal{T}) = ?$

Planning:

I. $\mathbb{F}_q$-linear triang. cat.

II. K-linear ... for any comm. ring K

I

Let $\mathcal{T}$ be a $\mathbb{F}_q$-linear triang. cat. Assume $\forall X, Y \in \mathcal{T}$, $\dim\left(\bigoplus_{n\in\mathbb{Z}} Ext^n(X,Y)\right) < \infty$

Typical examples: $\mathcal{T} = \mathcal{D}^b(fd B)$, B fd alg. / $\mathbb{F}_q$ with $gl.dim\, B < \infty$ or $\mathcal{T} = \mathcal{D}^b_{perf}(Coh\, X)$ where X is proper / $\mathbb{F}_q$ (if X proper + smooth then $\mathcal{D}^b_{perf}(Coh\, X) = \mathcal{D}^b(Coh\, X)$)

The analogies: abelian cats. $\longleftrightarrow$ triang. cats.

s.e.s. $\longleftrightarrow$ triangles, and so $H(\mathcal{T})$ has something to do with counting triangles.

Then the <u>derived Hall alg. of $\mathcal{T}$</u> is, as a $\mathbb{Q}$-vector space

$$DH(\mathcal{T}) = \mathbb{Q}\text{-vector space freely gen. on } \mathcal{X}_{\mathcal{T}} (= \mathcal{T}/\cong) = \{\chi_X\}_{X\in\mathcal{X}_{\mathcal{T}}}$$

We want a multiplication:

$$\chi_x \cdot \chi_y = \sum_z g^z_{xy} \, \chi_z \quad \text{where } g^z_{xy} \in \mathbb{Q} \text{ are the Hall numbers of } \mathcal{T}.$$

For abelian cats. we had

$$g^z_{xy} = \left| \{ 0 \to x' \to z \to y' \to 0 \mid x' \cong x, y' \cong y \} \right| = \frac{P^z_{xy}}{A_x A_y}$$

For triang. cats. $g^z_{xy} = \left| \{ x' \to z \to y' \xrightarrow{+} \mid x' \cong x, y' \cong y \} \right|$ but it gives a non-associative mult!

<u>Def:</u> The <u>Hall numbers of $\mathcal{T}$</u>

naive — correcting term

$$g^z_{xy} := \frac{|[x,z]_y|}{|\mathrm{Aut}(x)|} \cdot \prod_{i \geq 0} \frac{|[x, z[-i]]|^{(-1)^i}}{|[x, x[-i]]|^{(-1)^i}}$$

where $[x,z]_y = \{ f \in [x,z] \mid \mathrm{Cone}(f) = y \} \subseteq [x,z] = \mathcal{T}(x,z)$

<u>Thm:</u> $DH(\mathcal{T})$ defined like this is an associative algebra.

<u>Extra assumption</u>: $\mathcal{T} = \underbrace{\mathcal{D}_{fd}(B)}_{\text{derived cat. of dg } B\text{-mods. } E \text{ with } \dim H^\bullet(E) < \infty}$ where B is a dg algebra$/\mathbb{F}_q$

<u>Rk:</u> 1) This extra assumption is not needed for the thm (Xiao-Xu)

2) This ——— is always satisfied in practice. Later things will depend on B

To understand: in which sense g^2_{xy} counts triangles? Or, more important, in which sense $\prod_{i\geq 0} |[x,z[-i]]|^{(-1)^i}$ counts the morphisms $x \to z$?

Main idea (goes back to Quillen): which is the classifying space of $\mathcal{I}$?

Naive answer: if $\mathcal{C}$ is a categ., $B\mathcal{C}$ a top. space s.th. $\pi_0(B\mathcal{C}) \cong \mathcal{C}/\cong$,

$\pi_0(B\mathcal{C}, x) = \mathrm{Aut}(x)$, $\pi_i(B\mathcal{C}, x) = 0$ $\forall i \geq 2$, $\forall x$. (due essentially to Quillen)

But in the derived setting there is a new naive answer: if $\mathcal{I} = \mathcal{D}_{fd}(B)$,

Thm $\exists$ $\mathcal{X}(B)$ a top. space s.th. $\pi_0(\mathcal{X}(B)) \cong \mathcal{I}/\cong$, $\pi_1(\mathcal{X}(B), x) \cong \mathrm{Aut}_{\mathcal{I}}(x)$,

$\pi_i(\mathcal{X}(B), x) \cong [x, x[1-i]]$ $\forall i \geq 2$, $\forall x$

Rk) It is possible to show that $\mathcal{X}(B)$ is a model for the ∞-groupoid obtained by "inverting every arrow" $B\text{-dgmod}^?_{fd} = \{\text{quasi-isom. between fd dg } B\text{-modules}\}$

2) The usual groupoid completion of $B\text{-dgmod}^?_{fd}$ is the groupoid of objects in $\mathcal{I}$.

Def: The classifying space of objects in $\mathcal{I}$ is $\mathcal{X}(B)$ (depends on B!)

Def: The classifying space of morphisms in $\mathcal{I}$ is $\mathcal{X}^{(0)}(B)$ where

$\mathcal{X}^{(1)}(B)$ = same def but with $(B\text{-dgmod}^?_{fd})^I$ = cat. $\begin{array}{ccc} x & \to & y \\ \downarrow & & \downarrow \\ x' & \to & y' \end{array}$ morphism in B-dgmod

$$X^{(1)}(B) \underset{\text{target}}{\overset{\text{source}}{\rightrightarrows}} X(B) \dashrightarrow X^{(1)}(B) \dashrightarrow X(B)\times X(B)$$

Def: The classifying space of morphisms between fixed objects $x,y \in X$ is

$$\underline{\mathrm{Map}}(x,y) := \text{homotopy fiber of } X^{(1)}(B) \longrightarrow X(B)\times X(B) \text{ at } (x,y)$$

$\pi_0(\underline{\mathrm{Map}}(x,y)) \cong [x,y]$, $\pi_i(\underline{\mathrm{Map}}(x,y)) \cong [x,y[-i]]$, $i \geq 1$

Now, we want $|X_\cdot| \overset{?}{=} \sum_{x \in \pi_0(X)} \prod_{i>0} |\pi_i(X,x)|^{(-1)^i}$ for any hpy. space X with $|\pi_0(X)| < \infty$. This would implies

$$|\underline{\mathrm{Map}}(x,y)| = \prod_{i\geq 0} |[x,y[-i]]|^{(-1)^i}$$

Ex: If $\pi_i(X,x)=0$ $\forall i>1$, $|X| = \sum_{x\in\pi_0(X)} \frac{1}{|\pi_1(X,x)|} =$ orbifold number of points in $\pi_1(X)$ (= fundamental groupoid of X)

finite htpy types

FHT = (weak equivalence classes of hpy. spaces X with $|\pi_0(X)| < \infty$)

A measure of FHT (with coeff. in a ring R) is μ : FHT $\longrightarrow R$ s.th. $\mu(*)=1$, $\mu(\emptyset)=0$, $\mu(X \amalg Y) = \mu(X)+\mu(Y)$ and, if $X \xrightarrow{f} S$ is a map of objects in FHT with $\pi_0(S)=*$ then $\mu(X)=\mu(S)\cdot\mu(F)$ for F = (htpy fiber of f)

Prop: $\exists$ universal $\mu_0 : FHT \longrightarrow \mathbb{R}$, $X \longmapsto \sum_{x \in \pi_0(X)} \prod_{i \geq 0} |\pi_i(X,x)|^{(-1)^i}$ and

(any other such a map factors through to μ_0).

Now, $x, y \in \mathcal{T}$, $\underline{Map}(x,y) \in FHT$ and

$$\mu_0(\underline{Map}(x,y)) = \prod_{i \geq 0} |[x, y[-i]]|^{(-1)^i}$$

This is how we get

$$g^z_{xy} = \frac{|[x,z]_y|}{|Aut(x)|} \cdot \prod_{i>0} \frac{|[x, z[-i]]|^{(-1)^i}}{|[x, x[-i]]|^{(-1)^i}}$$

n° of $x \xrightarrow{f} z$ with Cone $f \cong y$

n° of automorphisms of x

Hence g^z_{xy} is $|\{x' \to z \to y' \xrightarrow{+} \mid x' \cong x, y' \cong y\}|$

Next lecture: suppose $\mathcal{T}$ is a $\mathbb{C}$-linear triang. cat. We still have

$X(B)$ classifying space for objects

$X^{(1)}(B)$ morphisms

$\underline{Map}(x,y)$ space with $\pi_i(\underline{Map}(x,y)) = [x, y[-i]]$

We have that $X(B)$ has an algebraic structure (it's a higher Artin stack for aff. top/$\mathbb{C}$)

$\rightsquigarrow g^z_{xy}$ has a motive $\in K_0(\text{Varieties})$.

22/11/2006,

Remainders: - $\mathcal{I}$ $\mathbb{F}_q$-linear triangulated cat.

- B $\mathbb{F}_q$-alg algebra ("finitely presented"), $\mathcal{I} \simeq \mathcal{D}^b(B)$

- $DH(\mathcal{I}) :=$ $\mathbb{Q}$-v.space on $\mathcal{X}_{\mathcal{I}}(= \mathcal{I}/\simeq)$

- $\chi_x \cdot \chi_y = \sum_z g_{xy}^z \cdot \chi_z$ where $g_{xy}^z := \frac{|[x,z]_y|}{|Aut(x)|} \prod_{i>0} \frac{|[x,z[-i]]|^{(-1)^i}}{|[x,x[-i]]|^{(-1)^i}}$

- Thm: $DH(\mathcal{I})$ is an associative $\mathbb{Q}$-algebra

- $\mathcal{X}(B)$ = classifying space of objects of $\mathcal{I}$, i.e. a dgp space with

$$\pi_0(\mathcal{X}(B)) \simeq \mathcal{X}_{\mathcal{I}}$$
$$\pi_1(\mathcal{X}(B), x) \simeq Aut(x)$$
$$\pi_i(\mathcal{X}(B), x) \simeq [x, x[1-i]]$$

- $\mathcal{X}^{(1)}(B)$ = classifying space of conflexions of $\mathcal{I}$

$$\begin{array}{ccc} \mathcal{X}^{(1)}(B) & \xrightarrow{c} & \mathcal{X}(B) \\ \pi \downarrow & & \\ \mathcal{X}(B) \times \mathcal{X}(B) & & \end{array}$$

when $\pi(x \xrightarrow{u} z) = (x, cone(u))$

$c(x \to z) = z$

Let $\mathcal{X}^{(1)}_{(x,y)}(B) \subseteq \mathcal{X}^{(1)}(B)$ the connected component over $(x,y) \in \mathcal{X}(B) \times \mathcal{X}(B)$

let $X^{(n)}_{x,y}(B) \xrightarrow{c} X(B)$ be the homotopy fiber of c at z in $X(x,y;z)$

$X(x,y;z) \in FHT$ = fin. homo. type

Prop: $F^t_{xy} = \#X(x,y;z)$ (pt of the local talk)

From this we can prove the thm. using homotopy theory.

Next step: $\mathcal{T}$ K-linear triang. category, K comm. ring, $\mathcal{T} \simeq \mathcal{D}_{pf}(B)$
B is a "fin presented" K-dg alg
full subcat. of dg B-modules which are perfect in CK

Main problem: define Hall numbers g^z_{xy} in this situation

$X(B)$, $X^{(n)}(B)$, $X(x,y;z)$ still exist but there are not finite homot. types anymore

Idea: use alg. geom $X(B)$, $X^{(n)}(B)$, $X(x,y;z)$ are closed to be alg. varieties / K
finite type alg variety

If $X(x,y;z)$ were an alg. variety of finite type then $[X(x,y;z)] \in K_0(Var) =$
$= (\mathbb{Z}$-module free over isoclasses of varieties$)$ // $([Y] = [Z] + [Y-Z]$ for $Z \hookrightarrow Y$ closed$)$
is a ring, denoted by R_{mot}, the ring of motivic numbers.

But $X(B), X^{(1)}(B) \ldots X(x,y,z)$ are not alg. varieties

"The underlying set of an alg. var. is a set" But $X(B), X^{(1)}(B), X(x,y,z)$ are not set but topological spaces! So, they can not underly the structure of an alg. variety.

Again, $\pi_0(X(B)), \pi_0(X^{(1)}(B)) \ldots$ are not alg. var.

$\pi_0(X(B)) = \mathcal{X}_{\mathcal{T}}$ and objects in $\mathcal{T}$ have non-trivial automorphisms, and so the moduli space doesn't exist as an alg. var. (think of $\mathcal{M}_g$ = curves of genus g).

There is the notion of coarse moduli space:

$$\pi_0(X(B)) \longrightarrow \mathcal{M} \text{ variety approximation}$$

But it doesn't work, since it's like going back to naive Hall numbers, without correcting terms: we're forgetting important information.

Conclusion: we need more technology than scheme t°. We need higher algebraic stacks

Stacks: is a generalization of sheave t° where you replace sets by homotopy types (i.e. spaces/htpy)

Def: k-Aff is the site of affine k-schemes + étale topology

We have $k\text{-Aff} = (k\text{-comm alg.})^{op}$

Def: A (∞) prestack is a foncteur $F : K\text{-Aff}^{op} \longrightarrow Top$ (sets are viewed in Top as discrete top. spaces). A strict morphism of prestacks F, G is a nat transformation $f : F \longrightarrow G$. A strict morphism is an equivalence if $\forall x \in K\text{-Aff}$, $F(x) \longrightarrow G(x)$ is a weak eq. of top. spaces. Define

$$Ho(PrSt(K)) := (equiv)^{-1} PrSt(K).$$

RK: PSh of sets on $K\text{-Aff}^{op}$ is the full subcat. $\{F \mid F(x)$ is weakly equiv. to a set$\} \subseteq Ho(PrSt(K))$.

$F \in Ho(PrSt(K))$ is a stack if $\forall x \in K\text{-Aff}$, $\forall U_\bullet \longrightarrow X$ étale hypercover we have $F(X) \xrightarrow{\sim} \operatorname{holim}_{m} F(U_m)$ (it is the homotopy analogue of sheaf cond.)

$$Ho(St(K)) := \{F \in Ho(PrSt(K)) \mid F \text{ is a stack}\}$$

RK: 1) $K\text{-Schemes} \hookrightarrow Ho(St(K)) \hookleftarrow \{$Artin stacks$\}$

$$K\text{-Schemes} \hookrightarrow Sh(K\text{-Aff}) \hookrightarrow Ho(St(K))$$

2) The l'a of stacks behaves like the theory of sheaves : (ho) limits, (ho) colimits, associated stacks, effectivity of eq. relat° ... quotients, ... but up to homotopy.

Example: $\mathcal{M}_J$ = moduli stack of objects in J (will depend on B).

$\mathcal{M}_J : K\text{-Aff}^{op} = (K\text{-comm alg}) \longrightarrow Top$

$$\begin{array}{ccc} K' & \longmapsto & X(B \overset{\mathbb{L}}{\otimes}_K K') = \text{classifying space of objects of } \mathcal{D}_{fd}(B \overset{\mathbb{L}}{\otimes}_K K') \text{ a } K'\text{-linear dg.} \\ \downarrow & & \downarrow \\ K'' & \longmapsto & X(B \overset{\mathbb{L}}{\otimes}_K K'') \end{array}$$

Prop: $\mathcal{M}_J$ is a stack, i.e. $\mathcal{M}_J \in Ho(St(K))$.

Geometric or algebraic stacks: $F \in Ho(St(K))$ is a geometric 0-stack if $\exists X$ alg. space over K s.t. $F(Z) \simeq \underbrace{Hom(Z, X)}_{\text{with the discrete topology}}$ functorial in Z.

Suppose that geometric $(n-1)$-stacks are defined, and smooth morphisms between them. F is a geometric n-stack if $\exists$ groupoid object $F_1 \underset{b}{\overset{s}{\rightrightarrows}} F_0$ in $(n-1)$-stacks s.t. s and b are smooth, and $F \simeq [F_0/F_1]$ in $Ho(St(K))$.

F is locally geometric if $F = \bigcup_i F_i$ if $F_i \hookrightarrow F$ open immersion and F_i is geometric n-stack (n depends on i).

Thm (jt. T. Vaquié) $\mathcal{M}_T$ is locally geometric and locally of finite presentation $/k$.

Pf: Hard and/or long. ✓

Rk: Basic notion of alg. geom. can be extended to locally geom. stacks.

Cor: $k = \overline{k}$ field, $x,y,z \in |\mathcal{T}|$, $[x,z]_y \subseteq [x,z]$.
is constructible

Cor: $[x,z]_y / \mathrm{Aut}(x) =: V(x,y,z)$ is a finite union of alg. var. $/k$ of finite type.

Cor $x,y,z \in |\mathcal{T}|$. $\mathcal{M}_{\mathcal{T}}(x,y) \subseteq \mathcal{M}^{(1)}_{\mathcal{T}} \longrightarrow \mathcal{M}_{\mathcal{T}}$

$\downarrow$

$\mathcal{M}_{\mathcal{T}} \times \mathcal{M}_{\mathcal{T}}$

$\mathcal{M}(x,y,z)$ = htpy fiber of $\mathcal{M}_{\mathcal{T}}(x,y)$ at z = moduli stack of triangles $x' \to z \to y' \to$ s.t. $x' \simeq x, y' \simeq y$

is a geometric n-stack for some n of finite presentation $/k$.

23/11/2006

Some references: Hall alg. for triang cats. / $\mathbb{F}_q$: math. QA/0504313

Stacks : alg-geom/9609014 (Simpson)

math. AG/0411037 } Theory of geom. ∞-stacks
math. AG/0604504 }

math. AG/0509098 $K_0(\infty\text{-stacks})$

math. AG/0503269 $\mathcal{M}_T$

Recall: $T = \mathcal{D}_{fd}(B)$, B dg alg. / K "fin. pres." K comm. ring

- $\mathcal{M}_T : K\text{-Aff}^{op} \longrightarrow Top$
  $K' \longmapsto X(B \overset{L}{\otimes}_K K')$
- $\mathcal{M}_T^{(1)} : K\text{-Aff}^{op} \longrightarrow Top$
  $K' \longmapsto X^{(1)}(B \overset{L}{\otimes}_K K')$

are locally geom. and locally fin. pres.

$\downarrow$

$X(x,y,z)$ = stacks of triangles $x' \to z \to y' \xrightarrow{+}$ with $x' \cong x$, $y' \cong y$

$\downarrow$

groupoïc n-stack of fin. presentation

- $R_{mot} := K_0(Var/k) = \left(\text{Free } \mathbb{Z}\text{-mod. on } K\text{-schemes of fin pres } /k\right) \Big/ \left(\begin{matrix} [Y] = [Z] + [Y - Z] \\ \forall Z \hookrightarrow Y \text{ closed} \end{matrix} \right)$

Problem: to define the class $[X(x,y,z)] \in R_{mot}$, or at least some localization of the ring R_{mot}, $R_{mot}[f^{-1}]$.

1. Grothendieck ring of geometric stacks

Notation: if F is a geom. n-stack, k a k-field, $x \in F(k)$, define

- $\pi_1(F,x): k\text{-Aff}^{op} \longrightarrow$ Groups + stratification
  $(k \to A) \longmapsto \pi_1(F(A),x)$

- $\pi_i(F,x)$: $i>1$ + $\pi_i(F(A),x)$

These explains the following:

Def: F is *special* if it is a geom. n-stack for some $n \geq 0$, it is fin pres. $/k$ and $\forall k, \forall x \in F(k)$, $\pi_1(F,x)$ is a linear alg. group $/k$
$\pi_i(F,x)$ is a unipotent fin alg. gr. $/k$, $i>1$
(e.g. $\pi_1(F,x)$ is a v. space $/k$)

Fact: $X(x,y,z)$ is special.

Def: $K_0(St^{sp}(k)) = ($free $\mathbb{Z}$-mod. over equiv. classes of special stacks$)$ / relations:

- $\forall F_0 \hookrightarrow F$ closed immersion $[F] = [F_0] + [F - F_0]$
- $\forall F_0$ special, $\forall F \xrightarrow{\delta} F'$ which is a Zariski locally trivial fibration with fiber F_0, $[F] = [F' \times F_0]$.

There is a map of rings (with direct products):

$$R_{mot} := K_0(Var/k) \longrightarrow K_0(St^T(k)) \quad , \sqrt[pvt]{\mathbb{L} := [\mathbb{A}^1] \in K_0(Var/k)}$$

Thm: The induced map

$$R_{mot}[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1}_{i>0}] \xrightarrow{\sim} K_0(St^{ir}(k))[\mathbb{L}^{-1}, (\mathbb{L}^i-1)^{-1}_{i>0}]$$

is an isomorphism.

Hence, "up to isom., inverting $\mathbb{L}$ and $\mathbb{L}^i - 1$, any special stack can be decomposed into k-schemes".

Def: $x, y, z \in \mathbb{Y}$, the absolute Hall number is

$$\tilde{g}^z_{xy} := [X(x,y,z)] \in R_{mot}[\mathbb{L}^{-1}, (\mathbb{L}^i-1)^{-1}_{i>0}]$$

Example: $k = \mathbb{F}_q \Rightarrow \exists$ ring morph $\#(?)(\mathbb{F}_q): R_{mot} \longrightarrow \mathbb{Z}$ inducing
$[X] \longmapsto \#X(\mathbb{F}_q)$

the morph. $R_{mot}[\mathbb{L}^{-1}, (\mathbb{L}^i-1)^{-1}_{i>0}] \longrightarrow \mathbb{Z}[q^{-1}, (q^i-1)^{-1}_{i>0}]$

Prop: $\#(\tilde{g}^z_{xy})(\mathbb{F}_q) = g^z_{xy}$ (cf. Lecture 1)

Example: $k = \mathbb{C} \Rightarrow \exists$ ring morph $F_H: R_{mot} \longrightarrow \mathbb{Z}[u,v]$ Hodge polyn. (class of X is smooth proj., extend else)
$[X] \longmapsto F_H([X]) = \sum_{i \geq 0} (-1)^i h^{p,q}(H^i_c(X)) u^p v^q$

and localizing one gets

$$F_{II} : K_0(St^{ST}(k)) \longrightarrow \mathbb{Z}[[u,v]][u^{-1},v^{-1}]$$

and so $F_{II}(\tilde{q}^2_{xy}) \in \mathbb{Z}[[u,v]][u^{-1},v^{-1}]$

In fact one has a morph for each cohom. theory giving interesting invariants.

2 Absolute Hall algebra

$x,y \in \mathcal{T}$, $\chi_x \cdot \chi_y = \sum_z \tilde{g}^z_{xy} \chi_z$ infinite and so replaced by $\int_{z \in \mathcal{M}_{\mathcal{T}}}$

We denote $\chi_x \cdot \chi_y = \int_{z \in \mathcal{M}_{\mathcal{T}}} \tilde{g}^z_{xy} = \mathcal{M}^{(1)}_{\mathcal{T}}(x,y) \longrightarrow \mathcal{M}_{\mathcal{T}}$ where the class of this mapped into

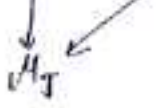
$\mathcal{M}^{(1)}_{\mathcal{T}}(x,y)$ the substack of $\mathcal{M}^{(1)}_{\mathcal{T}}$ consisting of $\{x \xrightarrow{u} z \mid x' \cong x$ for and $\mathrm{Cone}(u) \cong y\}$

and it is a special stack. Then $DH_{abs}(\mathcal{T})$ = relative version of $K_0(\mathcal{F} \longrightarrow \mathcal{M}_{\mathcal{T}})$

Def: $DH_{abs}(\mathcal{T})$ = free $\mathbb{Z}$-mod on equiv. class of $\mathcal{F} \longrightarrow \mathcal{M}_{\mathcal{T}}$, where $\mathcal{F}$ is special

+ relations: *) $[\mathcal{F}] = [\mathcal{F}_0] + [\mathcal{F} - \mathcal{F}_0]$ $\forall$ $\mathcal{F}_0 \hookrightarrow \mathcal{F}$ closed

**) $\mathcal{F} \xrightarrow{f} \mathcal{F}'$ and $\mathcal{F}_0 \longrightarrow \mathcal{M}_{\mathcal{T}}$ if f is a Zariski locally trivial with fiber $\mathcal{F}_0 \Rightarrow [\mathcal{F}] = [\mathcal{F}' \times_{\mathcal{M}_{\mathcal{T}}} \mathcal{F}_0]$



***) invert $\mathbb{A}^1 \times \mathcal{M}_{\mathcal{T}} \longrightarrow \mathcal{M}_{\mathcal{T}}$

$(\mathbb{A}^1 - 0) \times \mathcal{M}_{\mathcal{T}} \longrightarrow \mathcal{M}_{\mathcal{T}}$

Hall multiplications: $F_1 \longrightarrow \mathcal{M}_I$

$F_2 \longrightarrow \mathcal{M}_I$

$$\begin{array}{ccc} \mathcal{M}_I^{(2)} & \xrightarrow{c} & \mathcal{M}_I \\ \pi \downarrow & & \\ \mathcal{M}_I \times \mathcal{M}_I & & \end{array} \qquad \text{where} \quad \pi(x \xrightarrow{u} z) = (x, \mathrm{Cone}(u)), \quad c(x \to z) = z$$

$$[F_1] * [F_2] = \Big[(F_1 \times F_2) \times_{\mathcal{M}_I \times \mathcal{M}_I} \mathcal{M}_I^{(2)} \longrightarrow \mathcal{M}_I \Big] \in DH_{abs}(I)$$

Thm: $DH_{abs}(I)$ is an associative unital $R_{mot}\big[\mathbb{L}^{-1}, (\mathbb{L}^i - 1)^{-1}_{i>0}\big]$-algebra

Def: $DH_{abs}(I)$ is the absolute Hall algebra of I.

RK: $DH_{abs}(I)$ is non trivial, as one checks by using realizations in the motivic sense.

For instance, if

$Y = \mathbb{F}_q$: $DH(I)$ defined in Lecture 1

$\exists$ a surj. morp. of alg $DH_{abs}(I) \longrightarrow DH(I)$

$$\Big[F \to \mathcal{M}_I\Big] \longmapsto \sum_{x \in \pi_0(\mathcal{M}_I(\mathbb{F}_q))} \big(\#\, F_x(\mathbb{F}_q)\big) \cdot \chi_x$$

for F_x = fiber of $F \to \mathcal{M}_I$ at x

3. Open questions and/or research directions

1) Extra structures on $DH_{abs}(\mathcal{T})$: Green pairings, twisted version, ...

2) Very interesting/hard : The 2-periodic case: if $\mathcal{T} = \mathcal{D}^{(2)}_{fct}(B)$,

$DH(\mathcal{T}|_{\mathbb{F}_q}) = ?$, $DH_{abs}(\mathcal{T}) = ?$, ...

3) Relate $DH_{abs}(\mathcal{T})$ with $H^\bullet(\mathcal{M}_\mathcal{T})$ (question of Tom Bridgeland)

when $K = \mathbb{C}$

$\exists$ a map $DH_{abs}(\mathcal{T}) \dashrightarrow H^\bullet(\mathcal{M}_\mathcal{T})$ very singular and bad space!

$[F \xrightarrow{p} \mathcal{M}_\mathcal{T}] \mapsto [p_*(F)]$?

Probably, it's possible to define $DH_{abs}(\mathcal{T}) \dashrightarrow G_0(\mathcal{M}_\mathcal{T})[[t]][t^{-1}]$

$[F \xrightarrow{p} \mathcal{M}_\mathcal{T}] \mapsto p_!(\mathcal{D}_F) \mapsto \sum (-1)^p [Gr^p_H(p_!(\mathcal{D}_F))] t^p$

Hodge module

On nice substacks $\mathcal{M}_\mathcal{T}^{ss,\alpha} \subset \mathcal{M}_\mathcal{T}$ we can compose $G_\bullet(\mathcal{M}_\mathcal{T}^{ss,\alpha}) \xrightarrow{\chi} H_\bullet(\mathcal{M}_\mathcal{T}^{ss,\alpha})$.