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Homotopical and Higher Categorical Structures in Algebraic Geometry

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Mémoire de Synthèse
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Je dédie ce mémoire à Beussa la mignonne.

Pour ce qui est des choses humaines, ne pas rire, ne pas pleurer, ne pas s'indigner, mais comprendre.

Spinoza

Ce dont on ne peut parler, il faut le taire.

Wittgenstein

Ce que ces gens là sont cryptiques !

Simpson

Avertissements

La présent mémoire est une version étendue du mémoire de synthèse d'habilitation originellement rédigé pour la soutenance du 16 Mai 2003. Le chapitre §5 n'apparaissait pas dans la version originale, bien que son contenu était présenté sous forme fortement résumé.

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0 Introduction

During my PhD. thesis I have been working on cohomology and K-theory of algebraic stacks. Since then, my research interests moved to different questions concerning homotopical and higher categorical structures related to algebraic geometry. These questions have been originally motivated on one side by the work of C. Simpson on non-abelian Hodge theory and on the other side by the reading of A. Grothendieck's *Pursuing stacks*. From them, I have learned that homotopical and higher categorical methods can be used in order to define and study interesting invariants of algebraic varieties, and it is the main purpose of this short mémoire to present my recent researches in that direction (several of these works are joint works). I make my apologizes to the reader for the rather informal point of view I have decided to follow, first of all because some of these works have not been written up and will be presented without proofs (that hopefully will appear elsewhere), but also because even when written up only few of them have been published. In any case, I have tried to give complete and precise definitions, statements and/or references, so that the reader can at least have a rough idea on the state of my research activities at the present time. I also apologize as several of these works are still in progress, and therefore the results I present in this text are probably not the most definitive: in my opinion the interesting work still has to be done. And finally, I apologize for my poor English (this was my last apologizes).

All along this work, I have also tried to show that the results of this mémoire are not at all independent from each others and that they all belong to the realm of *homotopical mathematics*[1]. Mathematics are based on sets theory and the notion of structures (in Bourbaki's sense), while in homotopical mathematics sets are replaced by homotopy types, and structures are then enriched over the homotopy theory of spaces (e.g. groups are then replaced by simplicial groups, categories by S-categories, presheaves by simplicial presheaves, algebras by dga's ...) . The general philosophy (which is probably quite old and I guess goes back to Boardman, Dwyer, Kan, Quillen, Thomason, Waldhausen, Vogt, ...), seems to be that a huge part of mathematics does possess interesting and useful extensions to the context of homotopical mathematics. Of course, several examples of extensions of fundamental notions have already been studied by many authors, as S-categories, simplicial presheaves, A_∞-algebras and A_∞-categories .... I like to consider the results of the present work as part of the possible extensions of algebraic geometry to the realm of homotopical mathematics, what we call together with Gabriele *homotopical algebraic geometry*. To be a bit more explicit let me give the following board in which I express various notions appearing in this work as extensions of well known notions to the homotopical mathematics context. I apologize for the rather artificial flavor of this comparison, but I hope it helps anyway to give a unity to the results of this mémoire.

[1] I have learned this expression from M. Kontsevich.

Mathematics	**Homotopical Mathematics**
Sets	*Simplicial sets, homotopy types*
Categories	*Segal categories, or model categories*
Categories of functors $\underline{Hom}(A,B)$	*Segal categories* $\mathbb{R}\underline{Hom}(A,B)$
The category of sets, Sets	*The Segal category, or model category, of simplicial sets, Top*
Abelian categories	*Stable Segal categories* ($Def.2.5$), *or stable model categories*
Quillen's $K-$*theory of abelian categories*	$K-$*theory of stable Segal categories,* (§2.1) *or Waldhausen's* $K-$*theory of model categories*
Presheaves, sheaves	*Prestacks, stacks* (§3.1)
Topos	*Segal topos, model topos* (§3.1)
Grothendieck's Π_1 *of a topos*	$Pro-$*homotopy type of a Segal topos* ($Def.3.7$)
Algebraic completion Γ^{alg} *of a group* Γ	*Schematization* $(X\otimes k)^{sch}$ *of a space* X ($Thm.6.2$)
Affine gerbes	*Segal affine gerbes* ($Def.5.11$) *or Schematic homotopy types* ($Def.6.1$)
Tannakian category	*Tannakian Segal category* ($Def.5.11$)
Algebraic geometry	*Derived algebraic geometry* (§4.2)
Algebraic geometry relative to a base monoidal category C	*Homotopical algebraic geometry over a base monoidal Segal category, or model category,* M (§4.1)

Before describing a short overview of the content of the present work I would like to mention that there are several ways of dealing with homotopy theory and therefore of *doing homotopical mathematics*, not all of them being equivalent. I therefore had to make a choice. It is probably not so easy to explain this choice

(though I will try in section 1), but I wish to mention that it has been on one hand a pragmatic choice (i.e. I have chosen the theory that seemed to me the best suited for my particular purpose) but also a purely personal and psychological choice (i.e. I have chosen the theory I liked the most and with which I felt the most comfortable). As the reader will see (and might have already noticed in the previous comparison board), I have decided to work with essentially two theories: model categories and Segal categories. No doubts that one could also work with other theories. No doubts also that one could not work with some of them. In some sense, one of the main slogan of this work is: *the combination model categories/Segal categories is wonderful*.

This work is organized in 6 sections, following a rather arbitrary splitting into different themes, and which does not reflect at all a chronological order.

In the very first section I present some general remarks about homotopy theory, and in particular I try to compare various theories and approaches. This part is supposed to explain why one should not work with a unique theory but rather with (at least) two at the same time. This part does not contain any original mathematics.

In the second section, I present some results concerning Segal categories in their own. These results (partially joint with D.-C. Cisinski, A. Hirschowitz, J. Kock, C. Simpson and G. Vezzosi) have been proved *on the way*, in the sense that there were not part of my main objective, and are sometimes quite unrelated to algebraic geometry. However, from my point of view they are interesting examples showing the real flavor of Segal categories, and how they can be useful. This section also includes a brief overview of the relations between Segal categories and model categories, which will be used all along this work.

Section 3 is devoted to the notion of *Segal topos*, a natural extension of the notion of topos to the Segal setting, and its application to an extension of Artin-Mazur's style homotopy theory (this part is mainly a joint work with G. Vezzosi). Originally, the notion of Segal topos appeared for the need of our work on *homotopical algebraic geometry*, but seems to us of independent interests.

The fourth section is concerned with *homotopical algebraic geometry*, a joint work with G. Vezzosi, still in progress. I have included a short overview of the main definitions. The main results are then some applications of the general formalism to the construction of certain *derived moduli spaces*, and to what we like to call *brave new algebraic geometry*, an algebraico-geometric point of view on stable homotopy theory. Because the formalism of "HAG" is so widely ramified and provide very natural settings for several other results of this mémoire, I personally like to consider it as the deep heart of my recent works.

In section 5 I have included the present state of some thoughts about a Segal version of Tannakian duality. This part is unfortunately highly conjectural, but I have tried to present precise definitions and conjectures which are all mathematically meaningful. In fact the formalism of Tannakian duality for Segal categories was my starting point of my interests in homotopical and higher categorical structures, and

it provides a point of view that explains much more naturally several constructions and statement (that can be proved by complete different methods). It has been a great source of inspirations for me during the last few years.

Section 6 is devoted to my work on homotopy types and Hodge theory. The first part of this work is around the *schematization problem* of posed by Grothendieck in [Gr1], for which I have proposed a solution based on the notion of *schematic homotopy types*. In a second part, I present some joint work with T. Pantev and L. Katzarkov in which we define a Hodge decomposition of the schematization of a smooth projective complex manifold extending all the previously defined Hodge structures on cohomology, rational homotopy and fundamental group. I also propose a Tannakian interpretation, using the notion of Tannakian Segal categories of the last chapter, of theses constructions.

Finally, in an appendix I have reproduced a letter to P. May, containing some general thoughts about higher categories as well as a short note in which I construct comparison functors between certain homotopy theories.

Notations and conventions

First of all, I have decided to ignore universes considerations, and I will assume implicitly that certain objects are *small* when required. I apologize for this choice but I hope it will help keeping the redaction as light as possible.

I will use the books [Ho, Hir] as references for model category theory. For a model category M I will use the notation $Map_M(x,y)$ for the simplicial mapping space between two objects x and y, as defined in [Ho, §5.2]. The homotopy category of M, i.e. the localization of M along its sub-category of equivalences, is denoted by $Ho(M)$.

For the notion and basic definitions of S-categories I refer to [Dw-Ka1, Dw-Ka2]. For an S-category T I will use the notation $\underline{Hom}_T(x,y)$ for the simplicial set of morphisms between two objects x and y of T, and $Ho(T)$ for its homotopy category. The simplicial localization of a model category M along its sub-category of equivalences W is simply denoted by $LM := L(M,W)$.

The references for the theory of Segal categories are [Hi-Si, Pe, To-Ve1]. For a Segal category A I will denote by $A_{(x,y)}$ the simplicial set of morphisms between two objets x and y, and by $Ho(A)$ the homotopy category of A. For two Segal categories A and B I will denote by $\mathbb{R}\underline{Hom}(A,B)$ the derived internal Hom between A and B (see for example [To-Ve1, Cor. 2.0.5]). By definition, Top is the Segal category of simplicial sets $Top := LSSet$.

We will avoid using the expression *functor* in the context of Segal categories, and will reserve this expression for usual category theory. Instead we will speak about *morphisms between Segal categories*, and such morphisms will implicitly be assumed to live in some $\mathbb{R}\underline{Hom}$, or in other words that we have performed some fibrant replacement.

1 Homotopy theories

An abstract homotopy theory consists of a category of objects C, together with a sub-category W of equivalences and through which objects of C have to be identified. There exist several approaches to deal with such a couple (C, W), and the purpose of this section is to present some of them and to explain their advantages as well as their limits. What I would like to stress out is that it is sometimes necessary to use more than one approach at the same time in order to have a complete understanding of the situation. More precisely I would like to explain through some pertinent examples that the combination of model category theory and Segal category theory is probably one of the most efficient way of doing homotopy theory. I hope this will justify my choice of using model categories and Segal categories as *homotopical mathematics analogs* of categories.

Localization

The very first approach to abstract homotopy theory is the localization construction which associates to a pair (C, W) a localized category $W^{-1}C$, obtained from the category C by formally inverting the morphisms of W (see [Ga-Zi]). This construction has been of fundamental importance, for examples in order to define the homotopy category of spaces as well as derived categories. It is however very brutal, and much too coarse in many contexts. A striking example is when $C = Cat$ is the category of categories and W consist of all categorical equivalences. Then $W^{-1}C$ is the category of categories and isomorphism classes of functors between them. As functors can have non-trivial automorphisms, clearly some important information (e.g. the automorphism groups of the various functors) is lost when passing to $W^{-1}C$. Another, deeper, example is the fact that Waldhausen K-theory is not an invariant of homotopy categories, even when endowed with their natural triangulated structures when they exist (see [Sch]). Other related troubles, as the non functoriality of cones in derived categories, or the fact that derived categories of sheaves in general do not form a stack (see e.g. the introduction of [Hi-Si]) also show that the localization construction rapidly reaches its limits.

Model category

The major advance in abstract homotopy theory has been the notion of model categories introduced by D. Quillen (who has been followed by many different authors introducing various modified versions). This notion had an enormous impact and still today it seems difficult to do serious homotopy theory without referring to model structures. D. Quillen noticed that when a pair (C, W) is furthermore structured and admits a model structure then its homotopy theory on one hand becomes much more easy to deal with and to describe, and on the other hand possesses several additional important properties. First of all, the localized category

$W^{-1}C$, which in general is far from being easy to describe in a useful way, possesses a very friendly presentation using homotopy classes of maps between cofibrant and fibrant objects. The second fundamental consequence of the existence of a model structure is the existence of homotopy limits, homotopy colimits and of mapping spaces, which are very important for many homotopical constructions. Furthermore, homotopy limits, homotopy colimits and mapping spaces also possess workable descriptions which allow to really deal with them. These consequences of the existence of a model structure are definitely non-trivial, and they show that the existence of a model structure is never an easy nor a formal result.

Another very nice fact about model categories is that they tend to be easily encountered and appear in many different contexts. This is probably due to their good stability properties. For example, very often, presheaves (or even sheaves) with values in a model category again form a model category. In the same way, monoids objects, groups objects (or more general abstract structures) in a model category again form a model category. Starting from two fundamental examples of simplicial sets and complexes over a ring, one constructs this way a lot of interesting model categories.

However, model category theory also has its own limits. Indeed, all homotopical invariants one is interested in a model category only depends on the notion of equivalences, and not on the whole model structure. Model categories are therefore too much structured objects, and tend to be a too much rigid notion for certain purpose. For example, given two model categories M and N, it does not seem to exist a reasonable model category of functors between M and N. In other words, contrary to category theory, model category theory is not an *internal theory*, and this can cause troubles when one is interested in model categories as objects in their own. A good example of such a situation is when one considers a model category M, a (fibrant) object $x \in M$, and the model category M/x of objects over x. Then, the simplicial monoid $aut(x)$ of self equivalences of x is expected to *act* on the model category M/x, and this action does not seem to be easy to describe inside the theory of model category itself. This shows that one is naturally lead to go beyond the theory of model category, though every one seem to agree that model categories do include all the examples one would like to study. From my point of view, the problem encountered with model categories is not that they are too coarse objects but on the contrary that they are over-structured objects.

Dérivateurs

For a pair (C, W) and a category I, one can construct a new pair (C^I, W_I), where C^I is the category of functors from I to C and W_I the sub-category of natural transformations in C^I which levelwise belongs to W. Therefore, to an abstract homotopy theory (C, W) and a category I one can construct a new abstract homotopy theory of I-diagrams (C^I, W_I).

The two categories $(W^{-1}C)^I$ and $W_I^{-1}C^I$ are very different, and in general the category $W^{-1}C$ alone does not determine the category $W_I^{-1}C^I$. This is precisely one of the reason why the localization construction $(C,W) \mapsto W^{-1}C$ is too crude, and why the standard homotopical constructions (e.g. homotopy limits and colimits, mappings spaces ...) do depend on strictly more than the localized category $W^{-1}C$ alone. The main idea of the theory of *Dérivateurs*, which seems independently due to A. Heller and A. Grothendieck, is that from an abstract homotopy theory (C,W) one should not only consider the localized category $W^{-1}C$, but all the localized category of diagrams $W_I^{-1}C^I$ for various index categories I. Precisely, one should consider the (2-)functor

$$\begin{array}{rccc} \mathbb{D}_{(C,W)}: & \underline{Cat}^{op} & \longrightarrow & \underline{Cat} \\ & I & \mapsto & \mathbb{D}_{(C,W)}(I) := W_I^{-1}C^I \\ & (f: I \to J) & \mapsto & f^*, \end{array}$$

from the (opposite) 2-category of categories to itself.

According to A. Heller and A. Grothendieck the right context to do homotopy theory is therefore the 2-category of 2-functors $\underline{Cat}^{op} \longrightarrow \underline{Cat}$, called the 2-category of *pré-dérivateurs*, and denoted by $\underline{PDer}$. *Dérivateurs* are then defined to be pré-dérivateurs satisfying certain additional property, as for example the fact that the pull-backs functors f^* possess right and left adjoint (see [He, Gr2, Ma1] for details). A fundamental fact is that when a pair (C,W) does admit a model structure then the associated pré-dérivateur $\mathbb{D}_{(C,W)}$ is a dérivateur (this is essentially the existence of homotopy limits and colimits in model categories).

The theory of dérivateurs has been quite successful for many purposes. From a conceptual point of view, the dérivateur associated with a model category is a more intrinsic object than the model category itself (e.g. it only depends on the notion of equivalence and not on the whole model structure), which furthermore contains a lot of the homotopy invariants of the model category (as for example homotopy limits and colimits). It has been used for example in order to state some kind of universal properties that were lacking for derived categories and for the homotopy theory of spaces (see [Ma1] for historical references). The theory also has the advantage of solving many of the problems encountered with the localization construction, as for example the non functoriality of cones in derived categories (this was apparently one of the motivations to introduce them). Finally, on the contrary with model categories it has the advantage of being an internal theory, in the sense that given two dérivateurs one can define a reasonable (pré-)dérivateur of morphisms between them.

Because of all of these nice properties the theory of dérivateurs seems at first sight to be the *right context for doing homotopy theory*. However, there exist homotopical constructions that does not factor through the theory of dérivateurs, and the fundamental reason is that dérivateurs form a 2-category, which for many purposes is a too coarse structure missing some important higher homotopical information. One can make for instance the same kind of remark as for model categories. Given a model

category M and x a (fibrant) object in M, the simplicial monoid $aut(x)$ is expected to act on the dérivateurs associated to M/x. However, as this dérivateur lives in a 2-category this action automatically factors through an action of the 1-truncation $\tau_1 aut(x)$, and therefore one sees that the dérivateurs associated to M/x does not see the higher homotopical information encoded in the action of the whole space $aut(x)$. This fact implies for example that the K-theory functor (in the sense Waldhausen) can not be reasonably defined on the level of pré-dérivateurs (see Prop. 2.17 and Cor. 2.18). As explained by the result Thm. 2.16, the theory of pré-dérivateurs is only a approximation *up to* 2-*homotopies* of a more complex object encoding higher homotopical data.

Simplicial localization

In [Dw-Ka1] B. Dwyer and D. Kan introduced a refined version of the localization construction $(C, W) \mapsto W^{-1}C$, which associates to a pair (C, W) an S-category (i.e. a category enriched over the category of simplicial sets) $L(C, W)$ whose category of connected component $Ho(L(C, W))$ is naturally isomorphic to $W^{-1}C$. As the localized category $W^{-1}C$ satisfies a universal property in the category of categories, the S-category $L(C, W)$ satisfies some universal property, *up to equivalence*, in the category of S-categories. One of the main result proved by B. Dwyer and D. Kan states that when the pair (C, W) has a model structure then the simplicial localization $L(C, W)$ can be described using the mapping spaces defined in terms of fibrant-cofibrant resolutions (see [Dw-Ka2]). This last result is an extension to higher homotopies of the well known description of the category $W^{-1}C$ in terms of homotopy classes of maps between fibrant and cofibrant objects.

The very nice property of the S-category $L(C, W)$ is that it seems to contain all of the interesting homotopical information encoded in the pair (C, W). For example, when (C, W) is endowed with a model structure, the mapping spaces as well as the homotopy limits and colimits can all be reconstructed from $L(C, W)$. Also, as shown by Thm. 2.2, the K-theory functor does factor through the theory of S-categories, which shows that $L(C, W)$ contains definitely more information than the localized category $W^{-1}C$ (even when endowed with its additional triangulated structure, when it exists). Actually, the S-category $L(C, W)$ almost reconstructs, in a sense I will not precise here, the model category (C, W).

However, once again the theory of S-categories is not well behaved with respect to categories of functors and suffers the same troubles than the theory of model categories. Indeed, it does not seem so easy to define for two S-categories a reasonable S-category of morphisms between them. Of course there exists a natural S-category of morphisms between two S-categories but one can notice very easily that it does not have the right homotopy type (for example it is not invariant under equivalences of S-categories). Actually, there exist well known conceptual reasons why the category of S-categories can not be enriched over itself in a homotopical meaningful manner (see [Hi-Si], the remark after Problem 7.2). The situation is therefore very

similar to the case of model categories, as S-categories seem to model all what we want but does not provide and internal theory. In any case, one advantage of the notion S-categories compare to model categories is that it is more intrinsic, and the S-category associated to a model category only depends on its notion of equivalence and not on the whole model structure. The theory of S-categories is also better suited than the theory of dérivateur as there does not seem to exists homotopical invariants of model categories that can not be reconstructed from its simplicial localization.

Segal categories

Segal categories are weak form of S-categories, in which composition is only defined *up to a coherent system of equivalences* (see [Hi-Si, Pe] for details). An S-category is in an obvious way a Segal category, and any Segal category is equivalent to an S-category. More generally, the homotopy theory of S-categories and of Segal categories are equivalent (see [Si1, §2]). However, the main advantage of Segal categories is that they do form an internal theory. Given two Segal categories A and B, there exists a Segal category of morphisms $\mathbb{R}\underline{Hom}(A,B)$ satisfying all the required properties (as the usual adjunction rule, invariance by equivalences ...). When applied to two S-categories A and B, the Segal category $\mathbb{R}\underline{Hom}(A,B)$ should be interpreted as the Segal category of *lax simplicial functors from A to B*. In conclusion, the theory of Segal categories is equivalent to the theory of S-categories, but does behave well with respect to categories of functors.

For any abstract homotopy theory (C,W) one has the S-category $L(C,W)$, which can be considered as a Segal category. In particular, for two model categories M and N, one can consider their simplicial localizations LM and LN (along the sub-categories of equivalences), and then consider the Segal category $\mathbb{R}\underline{Hom}(LM,LN)$. The Segal category $\mathbb{R}\underline{Hom}(LM,LN)$ precisely plays the role of the non-existing model categories of morphisms from M to N. For a (fibrant) object x in M one can also make sense of the action of the simplicial monoid $aut(x)$ on the Segal category $L(M/x)$ (the model category of Segal category is a simplicial model category, and therefore an action of a simplicial monoid on a Segal category makes perfect sense).

The conclusion is that the theory of Segal categories can be used in order to make constructions with model categories that can not be done inside the theory of model categories itself. Several examples of such constructions are given in this mémoire. It worth also mentioning that model categories and Segal categories essentially model the same objects. Of course, it is not true that any Segal category can be written as some LM for a model category M. However, any Segal category can be written as $L(C,W)$, where C is a full sub-category of a model category which is closed by equivalences, and W is the restriction of equivalences for the ambient model structure. This remarks shows that, at least on the level of objects, Segal categories and model categories are essentially the same thing. What Segal categories really bring are new, much more flexible and powerful, well behaved functoriality

properties.

Other approaches

There exist other theories that I will not consider in this text, but which definitely behaves in very similar manners than Segal categories, and for which one could make the same analysis. I am thinking in particular to the theory of Quasi-categories of A. Joyal (see [Jo]), and the theory of complete Segal spaces of C. Rezk (see [Re1]). Some comparison functors between these theories and the theories of S-categories and Segal categories are given in appendix B.

What shows the above overview of the various ways of doing homotopy theory is that the combination of model category theory and Segal category theory seems the most suited for many purposes. Of course, one could also try to use only Segal category theory, but it is sometimes useful to use model categories which in practice provide a much more friendly setting than Segal categories. Roughly speaking, model categories are used in order to do explicit computations and Segal categories are rather used in order to produce abstract constructions (though this way of thinking is rather artificial). A key result which allows to really do this in practice is the so-called *strictification theorem* of [To-Ve1, Thm. 4.2.1]. By analogy, one could compare the relations between model categories and Segal categories with the existing relations between Grothendieck sites and topoi (passing from a site to its associated topos being the analog of the simplicial localization construction). I personally like to think that writing a Segal category A as some simplicial localization LM for a model category M is very much like *choosing coordinates on* A, in the same way that one choose coordinates on an abstract manifold when one does local computations. The relationship between Segal categories and model categories will be described with more details in the next section.

2 Segal categories

In this first section I will present some results and constructions about Segal categories, or that use Segal categories in an essential way. They are only few examples of the kind of results one can obtain using Segal categories, and I am convinced that more interesting results could also be proved in the future.

2.1 Segal categories and model categories

An S-category is by definition a category enriched over the category of simplicial sets. Segal categories are weak form of S-categories were the composition is only defined *up to a coherent system of equivalences*. For the details on the theory of

S-categories and Segal categories we refer to [Dw-Ka1, Dw-Ka2, Hi-Si, Pe], and to [To-Ve1] for an short overview and notations. From a purely esthetic point of view, it is also useful to think of Segal categories as ∞-categories in which i-morphisms are invertible (up to $i+1$-isomorphisms) for any $i > 1$.

The reader should keep in mind that Segal category theory works in a very similar manner than usual category theory and that most (if not all) of the standard categorical notions can be reasonably defined in the Segal setting. Here follows a sample of examples of such (once again we refer to the overview [To-Ve1] for more details).

- As any S-category, any Segal category A possesses a homotopy category $Ho(A)$ (which is a category in the usual sense), having the same objects as A and homotopy classes of morphisms of A as morphisms between them. We also recall that given two objects a and b in a Segal category, morphisms between a and b in A form a simplicial set denoted by $A_{(a,b)}$.

  For a morphism of Segal categories $f : A \longrightarrow B$, one says that f is essentially surjective (resp. fully faithful) if the induced functor $Ho(f) : Ho(A) \longrightarrow Ho(B)$ is essentially surjective (resp. if for any two objects a and b in A the induced morphism $f_{(a,b)} : A_{(a,b)} \longrightarrow B_{(f(a),f(b))}$ is an equivalence of simplicial sets). Of course one says that f is an equivalence if it is both fully faithful and essentially surjective.

- The foundational result about Segal categories is the existence of a model structure (see [Hi-Si, Pe]). Segal categories are particular cases of Segal pre-categories, and the category of Segal pre-categories is endowed with a cofibrantely generated model structure. Every object is cofibrant, and the fibrant objects for this model structure are Segal categories, but not all Segal category is a fibrant object, and in general fibrant objects are quite difficult to describe. The model category is furthermore enriched over itself (i.e. is an internal model category in the sense of [Hi-Si, §11]). This implies that given two Segal categories A and B one can associate a Segal category of morphisms

$$\mathbb{R}\underline{Hom}(A, B) := \underline{Hom}(A, RB),$$

  where RB is a fibrant model for B and $\underline{Hom}$ denote the internal Hom's in the category of Segal pre-categories. From the point of view of ∞-categories, $\mathbb{R}\underline{Hom}(A, B)$ is a model for the ∞-category of (lax) ∞-functors from A to B. In general, the expression $f : A \longrightarrow B$ *is a morphism of Segal categories* will mean that f is an object in $\mathbb{R}\underline{Hom}(A, B)$. In other words we implicitly allows ourselves to first take a fibrant replacement of B before considering morphisms into B.

- There is a notion of *Segal groupoid*, which is to Segal category theory what groupoids are for category theory. By definition, a Segal category A is a Segal

groupoid if its homotopy category $Ho(A)$ is a groupoid in the usual sense (or equivalently, if any morphism in A is a homotopy equivalence).

Furthermore for any Segal category A, one can define its geometric realization $|A|$, which is the diagonal simplicial set of the underlying bi-simplicial of A (see [Hi-Si, §2], where $|A|$ is denoted by $\mathcal{R}_{geq0}(A)$). The construction $A \mapsto |A|$ has a right adjoint, sending a simplicial set X to its fundamental Segal groupoid $\Pi_\infty(X)$ (this one is denoted by $\Pi_{1,se}(X)$ in [Hi-Si, §2]). By definition, the set of object of $\Pi_\infty(X)$ is the set of 0-simplicies in X, and for $(x_0, \ldots, x_n) \in X_0^{n+1}$ the simplicial set $\Pi_\infty(X)_n$ is the sub-simplicial set of X^{Δ^n} sending the i-th vertex of Δ^n to $x_i \in X_0$. A fundamental theorem states that the constructions $A \mapsto |A|$ and $X \mapsto \Pi_\infty(X)$ provide an equivalence between the homotopy theories of Segal groupoids and of simplicial sets (see [Pe, §6.3]). This last equivalence is a Segal version of the well known equivalence between the homotopy theories of 1-truncated homotopy types and of groupoids.

- Given a Segal category A and a set of morphisms S in $Ho(A)$ (we recall that $Ho(A)$ is the homotopy category of A), one can construct a Segal category $L(A, S)$ by formally inverting the arrows of S. This construction is the Segal analog of the Gabriel-Zisman localization for categories. By definition, the Segal category $L(A, S)$ comes with a localization morphism $l : A \longrightarrow L(A, S)$ satisfying the following universal property: for any Segal category B, the induced morphism

$$l^* : \mathbb{R}\underline{Hom}(L(A, S), B) \longrightarrow \mathbb{R}\underline{Hom}(A, B)$$

  is fully faithful, and its essential image consists of morphisms $A \longrightarrow B$ sending morphisms of S into equivalences in B (i.e. isomorphisms in $Ho(B)$).

  When applied to the case where C is a category considered as a Segal category, the construction $L(A, S)$ described above coincides, up to an equivalence, with the simplicial localization construction of [Dw-Ka1]. From the ∞-category point of view this means that $L(C, S)$ is the *∞-category obtained from C by formally inverting the arrows in S*. The fact that this localization procedure produces ∞-categories instead of categories is, from my point of view, the deep heart of the relations between homotopy theory and higher category theory.

- Given a model category M, one can construct a Segal category $LM := L(M, W)$ by localizing M, in the Segal category sense, along its sub-category of equivalences W. This gives a lot of examples of Segal categories. Using the main result of [Dw-Ka3] the Segal categories LM can be explicitly describe in terms of mapping spaces in M. In particular, when M is a simplicial model category LM is equivalent to the simplicial category of fibrant-cofibrant objects in M. For the model category of simplicial sets we will use the notation $Top := LSSet$. The Segal category Top is as fundamental as the category of sets in category theory.

Given a Segal category A one can construct a Yoneda embedding morphism

$$h : A \longrightarrow \mathbb{R}\underline{Hom}(A^{op}, Top),$$

which is know to be fully faithful (this is the Segal version of the Yoneda lemma). Any morphism $A^{op} \longrightarrow Top$ in the essential image of this morphism is called *representable*. Dually, one has a notion of *co-representable morphism*.

- Given a morphism of Segal categories $f : A \longrightarrow B$, one says that f has a right adjoint if there exists a morphism $g : B \longrightarrow A$ and a natural transformation $h \in \mathbb{R}\underline{Hom}(A, A)_{(Id,gf)}$, such that for any two objects $a \in A$ and $b \in B$ the natural morphism induced by h

$$A_{(f(a),b)} \xrightarrow{g_*} A_{(gf(a),g(b))} \xrightarrow{h^*} A_{(a,g(b))}$$

is an equivalence of simplicial sets. This definition allows one to talk about adjunction between Segal categories. An important fact is that a Quillen adjunction between model categories

$$f : M \longrightarrow N \qquad M \longleftarrow N : g$$

gives rise to a natural adjunction of Segal categories

$$Lf : LM \longrightarrow LN \qquad LM \longleftarrow LN : Lg.$$

- Given two Segal categories A and I, one says that A has limits (resp. colimits) along I if the constant diagram morphism $A \longrightarrow \mathbb{R}\underline{Hom}(I, A)$ has a right adjoint (resp. left adjoint). This allows one to talk about Segal categories having (small) limits (resp. colimits), or finite limits (resp. colimits). In particular one can talk about fibered and cofibered square, final and initial objects, left and right exactness ....

- Existence of Segal categories of morphisms also permits to define notions of algebraic structures in a Segal category. For example, if A is a Segal category with finite limits, the Segal category of monoids in A is the full sub-Segal category of $\mathbb{R}\underline{Hom}(\Delta^{op}, A)$ consisting of morphisms $F : \Delta^{op} \longrightarrow A$ such that $F([0]) = *$ and such that the Segal morphisms $F([n]) \to F([1])^n$ are equivalences in A. One can also defines this way groups, groupoids, rings, categories ... in A.

- More advanced notions for Segal categories, as topologies, stacks and topos theory, or monoidal structures will be given in §3 and §5.

The previous list (highly non-exhaustive) of examples of standard constructions one can do with Segal categories is very much useful in practice as it allows to use Segal category theory as the category theory we have learned at school. However,

as fibrant objects and fibrant resolutions are extremely difficult to describe these constructions turn out to be quite hard to manipulate in concrete terms. This difficult is solved by the so-called *strictification theorems*, which stipulate that when the Segal categories involved are of the form LM for M a model category, all of these categorical constructions can be expressed in terms of standard categorical construction inside the well known world of model categories. The most important strictification theorem, concerning categories of diagrams, is the following.

Let M be a simplicial model category (in the sense of [Ho]), and T be an S-category. Let M^T be the category of simplicial functors from T to M. A morphism $f : F \to G$ in M^T will be called an equivalence if for any object $t \in T$ the induced morphism $f_t : F(t) \to G(t)$ is an equivalence in M. By the universal property of the localization construction, one defines a natural morphism of Segal categories

$$L(M^T) \longrightarrow \mathbb{R}\underline{Hom}(T, LM),$$

where the localization on the left is perform with respect to the above notion of equivalences in M^T.

Theorem 2.1 *(Hirschowitz-Simpson, [To-Ve1, Thm. 18.6]) Under the previous assumption, the natural morphism*

$$L(M^T) \longrightarrow \mathbb{R}\underline{Hom}(T, LM)$$

is an equivalence of Segal categories.

Theorem 2.1 has many important consequences. First of all, it is the key argument in the proof of the Yoneda lemma for Segal categories. Furthermore, the Yoneda lemma implies that any Segal category A is equivalent to a full sub-Segal category of $L(SSet^T)$, where T is an S-category equivalent to A^{op}. This implies that any Segal category can be represented up to equivalences by a full sub-S-category of fibrant-cofibrant simplicial presheaves on a category. In particular, Segal categories and model categories are very close, and from my point of view are essentially the same kind of objects (i.e. ∞-categories where i-arrows are invertible for $i > 1$). Another, very important, consequence of theorem 2.1 is that for a model category M the Segal category LM has all limits and colimits, and these can be concretely computed in terms of homotopy limits and homotopy colimits in M.

In conclusion, the localization functor L permits to pass from model categories to Segal categories for which many interesting and abstract categorical constructions are available (e.g. Segal categories of functors). The strictification theorem then stipulates that these constructions on the level of Segal categories do have model category interpretations which in practice allows to reduce problems and computations to model category theory (and therefore to standard category theory). This principle will be highly used all along this work but in an rather implicit manner,

and corresponding constructions in model category and Segal categories will always be identified in some sense: for a given situation I will simply used the most appropriate theory. I personally like to deal as much as possible with Segal categories for general constructions, but use model categories to provide explicit descriptions. This does not quite follow the general philosophy, which I agree with, that Segal category theory (or any equivalent theory) should at some point completely replace model category theory and provide a much more powerful and friendly setting. My personal feeling is that the theory of Segal category is very much at its starting point and still we are not totally confident with the kind of arguments and constructions that one is allowed to use. Model categories on the other hand as been highly studied since many years, and I have the feeling that keeping an eye on model category theory while dealing with Segal categories will help us to learn how the theory really works. In particular this way of proceeding should provide a whole list of arguments, manipulations and constructions one can safely used, which I hope will be part of the standard mathematical knowledge in the future much in the same way as category theory is today.

To finish this paragraph on Segal category theory let me mention the existence of higher Segal categories. I will not use this higher notion very often, but it will happen that the notion of 2-Segal categories is needed. Also, I will implicitly use the *change of n* constructions given in [Hi-Si, §2], and always consider that a Segal category is also in a natural way a 2-Segal category. For all details I refer to [Hi-Si] and [Pe].

2.2 K-Theory

In the sixties A. Grothendieck asked whether or not K-theory is an invariant of triangulated categories. This question has been studied by several authors (R. Thomason, A. Neeman ...) and we now know that the answer is negative: there is no reasonable K-theory functor defined on the level of triangulated categories (see [Sch]). In other words, given a Waldhausen category C, the homotopy category $Ho(C)$, even endowed with its triangulated structure when it exists, is not sufficient to recover the K-theory spectra $K(C)$ (or even the K-theory groups $K_n(C)$).

However, for a Waldhausen category C, one can consider its simplicial localization LC defined by Dwyer and Kan and which is a refinement of the homotopy category $Ho(C)$ (see [Dw-Ka1]). It is an S-category that can be considered as a Segal category, and under some conditions on C (e.g. when it is *good* in the sense of [To-Ve2], though it seems the result stays correct under much weaker assumptions) LC is enough to recover the K-theory spectrum $K(C)$.

Theorem 2.2 *(Toën-Vezzosi, [To-Ve2]) The K-theory spectrum $K(C)$ of a* good *Waldhausen category C can be recovered (up to equivalence) functorially from the Segal category LC.*

An important non-trivial corollary of the previous theorem is the following. For a Segal category A we let $aut(A)$ its simplicial monoid of self-equivalences (as defined for example as in [Dw-Ka1]). Concretely, $aut(A)$ is the classifying space of the maximal sub-Segal groupoid $\mathbb{R}\underline{End}(A)^{int}$ of the Segal category of endomorphisms $\mathbb{R}\underline{End}(A)$.

Corollary 2.3 *For a* good *Waldhausen category C the simplicial monoid $aut(LC)$ acts naturally on the Waldhausen K-theory spectrum $K(C)$.*

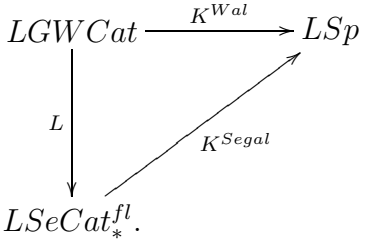
The proof of the above results given in [To-Ve2] is direct and does not involve Segal category techniques. However, as mentioned at the end of [To-Ve2], one could also prove theorem 2.2 by first defining a K-theory functor on the level of Segal categories having finite limits, and then proving that when applied to LC for a (good) Waldhausen category C the two constructions coincide. Without going into too technical details let us just mention the following unpublished result which follows from the results of [To-Ve2], the construction sketched at the end of [To-Ve2] and the strictification theorem of 2.1. For this, we let $GWCat$ be the category of good Waldhausen categories and exact functors. In $GWCat$, an exact functor $F : C \longrightarrow D$ is called an L-equivalence if the induced morphism $LF : LC \longrightarrow LD$ is an equivalence of Segal categories. We let $LGWCat$ be the Segal category obtained from $GWCat$ by applying the simplicial localization functor with respect to the L-equivalences. In the same way, let $LSeCat_*$ be the Segal category obtained from the category of pointed Segal categories [2] by applying the simplicial localization functor with respect to equivalences. We denote by $LSeCat_*^{fl}$ the sub-Segal category of $LSeCat_*$ consisting of Segal categories having finite limits and left exact functors. Finally, let LSp the simplicial localization of the model category of spectra. The Waldhausen K-theory functor induces a well defined morphism of Segal categories

$$K^{Wal} : LGWCat \longrightarrow LSp.$$

In the same way, the simplicial localization functor induces a morphism

$$L : LGWCat \longrightarrow LSeCat_*^{fl}.$$

Theorem 2.4 *There exists a commutative diagram in the homotopy category of Segal categories*



[2] A Segal category is pointed if it has an initial object which is also a final object.

The conclusion of theorem 2.4 really is:

K-theory is an invariant of Segal categories.

2.3 Stable Segal categories

In the last part, we have seen that the K-theory spectrum of a Waldhausen category can be recovered from its simplicial localization LC without any additional structures. At first sight, this might look surprising as several works around this type of questions involves triangulated structures (see [Du-Sh, Ne]). In fact, the S-category LC completely determines the notion of fiber and cofiber sequences in the homotopy category $Ho(C)$. In particular, the category $Ho(C)$ together with its triangulated structure (when it exists) is completely determined by the Segal category LC.

This observation has led A. Hirschowitz, C. Simpson and myself to introduce a notion of *stable Segal categories*. This notion clearly is very close to the notions of *enhanced triangulated categories* (see [Bo-Ka]), of *triangulated A_∞-categories* of M. Kontsevich, and of *stable model categories* (see [Ho, §7]).

Definition 2.5 *(Hirschowitz-Simpson-Toën) A Segal category A is* stable *if it satisfies the following three conditions.*

1. *The Segal category A possesses finite limits and colimits (in particular its a final object and an initial object).*
2. *The final and initial object in A are equivalent.*
3. *The suspension functor*

$$\begin{array}{rccc} S: & Ho(A) & \longrightarrow & Ho(A) \\ & x & \mapsto & * \coprod_x * \end{array}$$

is an equivalence of categories.

Remark 2.6 Though the notion of stable Segal category is not strictly speaking a generalization of the notion of abelian categories (a category which is stable in the Segal sense is trivial), stable Segal categories really play the role of abelian categories in the Segal setting.

The main properties of stable Segal categories are gathered in the following unpublished theorem.

Theorem 2.7 *1. The homotopy category $Ho(A)$ of a stable Segal category A has a natural triangulated structure, for which triangles are induced by the images of fiber sequences in A. Any exact morphism $f : A \longrightarrow B$ between stable Segal categories induces a functorial triangulated functor $Ho(f) : Ho(A) \longrightarrow Ho(B)$.*

2. *A morphism between stable Segal category is left exact if and only if it is right exact.*

3. *For any stable Segal category A and any Segal category B the Segal category $\mathbb{R}\underline{Hom}(B, A)$ is stable.*

4. *If C is a full sub-category of a stable model category M (in the sense of [Ho, §7]) which is closed by equivalences M and homotopy fibers and contains the initial-final object, then the Segal category LC is stable.*

5. *For any stable Segal category A, there exists a full sub-category C of a stable model category and which is closed by equivalences in M and homotopy fibers such that LC is equivalent to A. Furthermore, one can chose M to be a model category of presheaves of spectra over some category.*

The previous theorem clearly shows that stable Segal categories are quite close to triangulated categories. However, the additional structure encoded in Segal categories allows one to have the fundamental property (3), which is violated for triangulated categories. Furthermore, the theorem 2.2 together with the counter-example given in [Sch], show that there exists two non-equivalent stable Segal categories whose homotopy categories are equivalent as triangulated categories. This of course implies that a stable Segal category A contains strictly more information than its triangulated homotopy category $Ho(A)$. Because of all these reasons, we propose the notion of stable Segal categories as an alternative to the notion of triangulated categories, and we think that several troubles classically encountered with triangulated categories can be solved this way. This is a reasonable thing to do as all of the interesting triangulated categories we are award of (in particular all of triangulated categories of geometric origin, as for example derived categories of sheaves) are of the form $Ho(A)$ for some stable Segal category A. I personably tend to think that triangulated categories which are not of the form $Ho(A)$ for some stable Segal categories A are unreasonable object and should not be considered at all.

To finish, let me mention that the notion of stable Segal categories has been already used in several contexts, as for examples the Tannakian formalism for Segal categories (see §5) and the stacks point of view on Grothendieck's duality theory (Hirschowitz-Simpson-Toën, unpublished).

2.4 Hochschild cohomology of Segal categories

In this part I present some thoughts about the notion of Hochschild cohomology of Segal categories. My main objective was to understand in which sense *Hochschild cohomology is the space of endomorphisms of an identity functor*, as stated by several authors (see e.g. [Sei]). The point of view taken in this paragraph is non-linear, and is concerned with the discrete version of the Hochschild cohomology. Taking linear

structures into account is probably more complicated and actually a bit tricky, but I guess it should be doable.

All the material of this section is not written up, except the last result proved in collaboration with J. Kock (see [Ko-To]).

For a Segal category A one has its Segal category of endomorphisms $\mathbb{R}\underline{End}(A)$. One consider the object $Id \in \mathbb{R}\underline{End}(A)$ and its simplicial set of endomorphisms $\mathbb{R}\underline{End}(Id_A) := \mathbb{R}\underline{End}(A)_{(Id_A, Id_A)}$.

Definition 2.8 *The* Hochschild cohomology *of a Segal category A (also called the* center *of A) is the simplicial set*

$$\mathbb{HH}(\mathbb{A}) := \mathbb{R}\underline{End}(Id_A).$$

Remark 2.9 When applied to the Segal category LM for a model category M, definition 2.8 gives a notion of the *Hochschild cohomology of the model category M.*

This definition is a direct generalization of the center of a category C, defined as the monoid of endomorphisms of the identity functor of C. For a category C, its center is always a commutative monoid. This follows from a very standard argument as the center is always endowed with two compatible unital and associative composition laws. For a Segal category A, we will see that $\mathbb{HH}(A)$ is not quite commutative, but is a 2-*Segal monoid*, or in other words is endowed with two compatible unital and associative *weak* composition laws. Therefore, $\mathbb{HH}(A)$ looks very much like a 2-fold loop space but for which the composition laws are not necessarily invertible. In order to state precise results let me start by some definitions (the reader could consult [Ko-To] for details).

We let C be a category with a notion of equivalences (e.g. a model category or a Waldhausen category) and finite products, and such that finite products preserve equivalences. A *Segal monoid* in C is a functor

$$\begin{array}{rccc} H: & \Delta^{op} & \longrightarrow & C \\ & [n] & \mapsto & H_n, \end{array}$$

such that

1. $H_0 = *$
2. For any $n \geq 1$ the Segal morphism (see [Hi-Si] or [To-Ve1, §2])

$$H_n \longrightarrow H_1^n$$

is an equivalence.

A morphism between Segal monoids is simply a natural transformation, and a morphism is an equivalence if it is an equivalence levelwise (or equivalently on the image of [1]). Clearly, Segal monoids in C do form a category $SeMon(C)$, again with a notion of equivalences. Furthermore, $SeMon(C)$ has finite products which again preserve equivalences. The construction can therefore by iterated.

Definition 2.10 *Let C be a category with a notion of equivalences and finite products, and such that finite products preserve equivalences.*

1. *The category of* d-Segal monoids *in C is defined inductively by*

$$0 - SeMon(C) = C \qquad d - SeMon(C) := SeMon((d-1) - SeMon(C)).$$

   d-Segal monoids in simplicial sets are simply called d-Segal monoids.

2. *The underlying object of a Segal monoid A in C is $A_1 \in C$. Inductively, the underlying object of a d-Segal monoid A in C is the underlying object of the $(d-1)$-Segal monoid $A_1 \in (d-1) - SeMon(C)$. The underlying object of a d-Segal monoid A is again denoted by A.*

3. *The category of* d-fold monoidal Segal categories *is the category of d-Segal monoids in the category of Segal categories.*

Almost by definition, one has the following elementary proposition.

Proposition 2.11 *Let A be a d-fold monoidal Segal category, and still denote by A the underlying Segal category. The Hochschild cohomology of A, $\mathbb{HH}(A)$, has a natural structure of a $(d+2)$-Segal monoid.*

Remark 2.12

1. The expression *has a natural structure of a $(d+2)$-Segal monoid* is a bit ambiguous. It as of clearly to be understood up to equivalence. More precisely, this means that there exists a natural $(d+2)$-Segal monoid M whose underlying object is naturally equivalent to $\mathbb{HH}(A)$.

2. Of course, the $(d+2)$-Segal monoid structure on $\mathbb{HH}(A)$ depends on the d-fold monoidal structure given on A.

The above proposition becomes really interesting with the following theorem which relates the definition of Hochschild cohomology of a Segal category to a more usual one. Let us recall first that for a simplicial monoid H one has a notion of H-modules, or equivalently of simplicial sets with an action of H. The category of H-module is known to be a model category for which equivalences are defined on the underlying simplicial set. The main theorem, which is a direct consequence of the strictification theorem 2.1, is the following.

Theorem 2.13 *Let H be a simplicial monoid and $A = BH$ be the Segal category with one object and H as endomorphisms of this object. Then, one has a natural equivalence of simplicial sets*

$$\mathbb{HH}(BH) \sim Map_{H\times H^{op}-Mod}(H, H),$$

where the right hand side is the mapping space computed in the model category of $H \times H^{op}$-modules.

An important corollary is a non-linear analog of Deligne's conjecture on the complex of Hochschild cohomology of an associative differential graded algebra (see [Ko-To] for detailed references). It follows from proposition 2.11 and theorem 2.13 and by the observation that if H is a d-Segal monoid then BH is a naturally a $(d-1)$-fold monoidal Segal category (the notions of modules over simplicial monoids extend naturally to Segal monoids, using for example that any Segal monoid is naturally equivalent to a simplicial monoid).

Corollary 2.14 *Let H be a d-Segal monoid (in simplicial sets). Then the simplicial set $Map_{H\times H^{op}-Mod}(H, H)$ has a natural structure of a $(d+1)$-Segal monoid.*

By different techniques, which use in an essential way the simplicial localization functor of Dwyer and Kan one can also prove the following related result. It does imply corollary 2.14 for $d = 1$, but also has its own interest. It is a model category analog of the fact that the endomorphism of the unit in a monoidal category is a commutative monoid.

Theorem 2.15 *(Kock-Toën, [Ko-To]) Let M be a monoidal model category in the sense of [Ho, §4.3] and 1_M be its unit. Then the simplicial set $Map_M(1_M, 1_M)$ has a natural structure of a 2-monoid.*

2.5 Segal categories and dérivateurs

In this last paragraph I will compare the theory of *dérivateurs* of A. Heller and A. Grothendieck with the theory of Segal categories. The main theorem states that the theory of dérivateurs is essentially a 2-truncation of the theory of Segal categories, and so the two theories are more or less equivalent *up to* 2*-homotopies*. The results of this paragraph have not been written up.

We denote by $\underline{PDer}$ the 2-category of pré-dérivateurs in the sense of [Ma1]. Recall that $\underline{PDer}$ is the 2-category of 2-functors $\underline{Cat}^{op} \longrightarrow \underline{Cat}$. For a Segal category A we define an object $\mathbb{D}_A \in \underline{PDer}$ in the following way

$$\begin{array}{rccc} \mathbb{D}_A : & \underline{Cat}^{op} & \longrightarrow & \underline{Cat} \\ & I & \mapsto & Ho(\mathbb{R}\underline{Hom}(I, A)). \end{array}$$

The construction $A \mapsto \mathbb{D}_A$ clearly defines a morphism of 2-Segal categories

$$\mathbb{D} : \underline{SeCat} \longrightarrow \underline{PDer},$$

where the left hand side is the 2-Segal category of Segal categories as defined in [Hi-Si, S]. As the 2-Segal category $\underline{PDer}$ is a 2-category, this morphism factors as a 2-functor between 2-categories

$$\mathbb{D} : \tau_{\leq 2}\underline{SeCat} \longrightarrow \underline{PDer},$$

where we have denoted by $\tau_{\leq 2}B$ the 2-category obtained from a 2-Segal category by replacing all 1-Segal categories of morphisms in B by their homotopy categories ($\tau_{\leq 2}B$ is the 2-Segal analog of the homotopy category of Segal categories). The following theorem has been proved in collaboration with D.-C. Cisinski.

Theorem 2.16 *(Cisinski-Toën) The above 2-functor*

$$\mathbb{D} : \tau_{\leq 2}\underline{SeCat} \longrightarrow \underline{PDer}$$

is fully faithful (in the sense of 2-categories).

The above result implies that the theory of pré-dérivateurs is an approximation, *up to 2-homotopies*, of the theory of Segal categories. However, the functor $\mathbb{D}$ is surely not essentially surjective and I personably think that pré-dérivateurs not in the essential image of $\mathbb{D}$ are very unnatural objects which should not be considered at all.

I would like to finish this part by the proposition below showing that the higher homotopies that are not taken into account in the theory of pré-dérivateurs are of some importance. For this, we recall that there is a K-theory functor $K^{Wal} : LGWCat \longrightarrow LSp$, from the Segal category of good Waldhausen categories to the Segal categories of spectra (see §2.1).

Proposition 2.17 *Let n be any integer. The morphism of Segal categories*

$$K^{Wal} : LGWCat \longrightarrow LSp$$

does not factor, in the homotopy category of Segal categories, through any Segal category whose simplicial sets of morphisms are n-truncated.

The proof of this proposition relies on the fact that for a space X, the simplicial monoid of auto-equivalences $aut(X)$ acts naturally on its K-theory spectrum $K(X)$ (here $K(X)$ is the space of algebraic K-theory of X, as defined by Waldhausen), and in general this action does not factor through the n-truncation of $aut(X)$. One could also use the same kind of arguments using the action of the simplicial monoid of auto-equivalences $aut(A)$ of a simplicial ring A on its K-theory spectra $K(A)$.

An important consequence of the last proposition is the following corollary.

Corollary 2.18 *Waldhausen K-theory functor can not factor, up to a natural equivalence, through any full sub-2-category of $\underline{PDer}$.*

As a consequence of the corollary 2.18 we see that Waldhausen K-theory can not be reasonably defined on the level of *dérivateurs triangulés* of [Ma2]. However, this does not give a counter example to conjecture 1 of [Ma2] as it is only stated for the K-theory of an exact category and not for a larger class of Waldhausen categories (e.g. including Waldhausen categories computing K-theory of spaces or of simplicial rings). In the same way, Proposition 2.17 also implies that there is no reasonable K-theory functor defined on the level of triangulated categories.

The conclusion of proposition 2.17 really is:

Waldhausen K-theory is not an invariant of n-categories for any $n < \infty$.

3 Segal categories, stacks and homotopy theory

Segal categories are generalizations of categories, and *Segal topoi* are to Segal categories what Grothendieck topoi are to categories. The basic notions and results of the theory are presented in [To-Ve1] and [To-Ve3]. The notion of Segal topoi appeared naturally at the very beginning of my joint work with Gabriele Vezzosi, in our investigation of *homotopical algebraic geometry* ("HAG" for short). Indeed, a very natural setting for algebraic geometry is the category of sheaves of sets on the site of affine schemes, or more generally of 1-stacks, 2-stacks or even ∞-stacks (simply called *stacks* in this text). While developing the basic theory of HAG we discovered that the notion of stacks over Grothendieck sites is too restrictive for our purposes. Instead, a notion of *stacks over Segal sites* (i.e. a Segal category endowed with a suitable notion of Grothendieck topologies) were needed. As topoi are categories of sheaves, *Segal topoi are Segal categories of stacks over Segal sites.*

Despite its conceptual interest, the notion of Segal topoi turned out to appear in several contexts, and seem to be a natural and useful notion (see [La] for a surprising context of apparition). As an example of application we have investigated (still with Gabriele Vezzosi) a reinterpretation and a generalization of Artin-Mazur's homotopy type, which appear now as part of *higher topos theory*, in the same spirit as A. Grothendieck's considerations on homotopy types of topoi found in his letter to L. Breen (see [Gr1]).

3.1 Segal topoi

Definition 3.1 *1. Let A be a Segal category. A Segal category B is a* left exact localization of *A if it is equivalent to a full sub-Segal category B' of A such that the inclusion functor $A' \hookrightarrow B$ has a left exact left adjoint.*

2. *A Segal category A is a* Segal topos *if there exists a Segal category T such that A is a left exact localization of $\hat{T} := \mathbb{R}\underline{Hom}(T^{op}, Top)$.*

The above definition is based on the fact that a Grothendieck topos is a category which is a left exact localization of a category of presheaves of sets (see e.g. [Mac-Moe, Schu]). Let me mention immediately that there exists Segal topoi which are not left exact localization of $\hat{C}$ for a category C (see [To-Ve1, Rem. 2.0.7] for a counter-example). This implies that the fact that T is a Segal category in definition 3.1 (2) and not just a category is crucial. Actually, the Segal topoi appearing in HAG are not exact localization of $\hat{C}$ for a category C.

Definition 3.2 *A* Segal topology on a Segal category T *is a Grothendieck topology on the homotopy category $Ho(T)$. A Segal category together with a Segal topology is called a* Segal site.

When a Segal category T is endowed with a topology τ one can define a notion of hyper-coverings which generalizes the usual notion (see [To-Ve1, Def. 3.3.2 (1)]). More precisely, one says that a morphism $f : F \longrightarrow F'$ in $\hat{T}$ is a (τ-)epimorphism if for any object $t \in T$ and any $x \in \pi_0(F'(x))$, there is a covering sieve S of $t \in Ho(T)$ such that for any $u \to x$ belonging to S there exists $y \in \pi_0(F(u))$ with $f(y) = x_{|u}$. Now, a morphism $f : F \longrightarrow F'$ will be called a (τ-)hypercovering if for any integer $n \geq 0$ the natural morphism

$$F \longrightarrow F' \times_{(F')^{\partial\Delta^n}} F^{\partial\Delta^n}$$

is a ($\tau-$)epimorphism in $\hat{T}$. Here, F^K denotes the exponentiation of F by an object $K \in Top$, which is uniquely determined by the usual adjunction formula[3]

$$\hat{T}_{(G,F^K)} \simeq Top_{(K,\hat{T}_{(G,F)})}.$$

For an object $G \in \hat{T}$, we will say that F satisfies the descent condition for hypercoverings if for any hypercovering $F \longrightarrow F'$ in $\hat{T}$ the natural morphism

$$\hat{T}_{(F',G)} \longrightarrow \hat{T}_{(F,G)}$$

is an equivalence in Top. This decent condition is the Segal analog of the usual sheaf and the stack conditions.

Concerning the terminology, the Segal category of pre-stacks on a Segal site (T, τ) is the Segal category $\hat{T} = \mathbb{R}\underline{Hom}(T^{op}, Top)$, and the Segal category of stacks on (T, τ) is the full sub-Segal category of $\hat{T}$ consisting of pre-stacks having the descent property for hyper-coverings (see [To-Ve1, Def. 3.3.2 (2)]). The Segal category of stacks is denoted by $T^{\sim,\tau}$.

[3]Such exponentiation exists in any Segal category with arbitrary limits.

An important result states that the Segal category $T^{\sim,\tau}$ is the localization of a natural model category of stacks. Indeed, in [To-Ve3] is constructed a model category of stacks on the Segal site (T,τ), which is denoted by $SPr_\tau(T)$. It is a direct consequence of the strictification theorem 2.1 that $LSPr_\tau(T)$ is equivalent to $T^{\sim,\tau}$. In particular, one sees that the Segal category $T^{\sim,\tau}$ possesses all kind of limits and colimits. In practice the existence of this model structure provides a more friendly setting for stacks over Segal sites.

The main theorem of Segal topos theory is the following statement that relates Segal topoi and Segal categories of stacks. It is a generalization of the classical correspondence between Grothendieck topologies and exact localizations of categories of pre-sheaves. For this we need the extra notion of *t-complete Segal categories* for which we refer to [To-Ve1, Def. 3.3.6] for a detailed definition (a Segal topos is t-complete if every hyper-covering is contractible).

Theorem 3.3 *(Toën-Vezzosi, [To-Ve1, Thm. 3.3.8]) Let T be a Segal category.*

1. *For any topology τ on T, the inclusion morphism $T^{\sim,\tau} \hookrightarrow \hat{T}$ possesses a left exact left adjoint.*

2. *The map $\tau \mapsto T^{\sim,\tau}$, which associates to a topology τ the full sub-category of stacks on (T,τ) induces a bijective correspondence between topologies on T and t-complete full sub-Segal categories A on $\hat{T}$ which inclusion functor $A \hookrightarrow \hat{T}$ possesses a left exact left adjoint.*

3. *The t-complete Segal topoi as precisely the Segal categories of stacks over a Segal site.*

The interest of this last result lies in the fact that it justifies our notion of topologies on Segal categories, at least when one is dealing with t-complete Segal topoi. It seems possible however to drop the t-complete assumption by replacing the notion of topologies by a weaker notion of *hyper-topology*. Informally, in a topology one fixes the data of coverings whereas in an hyper-topology one fixes the data of hyper-coverings (it is worth mentioning here that even in the case of a category, the two notions of topologies and hyper-topologies do not coincide). We have not investigate this notion seriously as we did not find any reasons to work with Segal topoi which are not t-complete, and it is not clear at all that this new notion of hyper-topology has any interest besides a conceptual one.

Of course theorem 3.3 is only the starting point of the whole theory, and much work has to been done in order to have a workable and powerful theory of Segal topoi. Unfortunately we have not done much more, except stating a conjectural Giraud's style characterization of Segal topoi (see [To-Ve1, Conj. 5.1.1]) for which some recent progress have been made by J. Lurie (see [Lu1]).

To finish this paragraph, let me mention the unpublished work of C. Rezk on the notion of *homotopy topos* (I personally prefer the expression *model topos*) which is a model category analog of the notion of Segal topos (see [To-Ve3, Def. 3.8.1]). By definition, if M is a model topos then the Segal category LM is a Segal topos, and furthermore any Segal topos is obtained this way. However, the main advantage of Segal topoi compare to model topoi is the existence of a good notion of morphisms. In other words, given two Segal topoi T and T' there exist a Segal category of geometric morphisms $\mathbb{R}\underline{Hom}^{geom}(T,T')$, defined as the full sub-Segal category of $\mathbb{R}\underline{Hom}(T',T)$ consisting of exact morphisms which admit a right adjoint, which allows one to consider all Segal topoi together assembled in a 2-Segal category. These Segal categories of geometric morphisms will be used in a an essential way in the next paragraph.

3.2 Homotopy type of Segal topoi

The starting point of homotopy theory of Segal topoi has been the following Grothendieck's style interpretation of homotopy types of spaces. It is also a possible answer to some conjecture of Grothendieck that appear in one of his letter to L. Breen (see [To1] for comments on this conjecture). It uses the notion of Segal topoi and Segal categories of geometric morphisms between them. Before stating this result let me recall that given two Segal topoi T and T' there exists a Segal category of geometric morphisms $\mathbb{R}\underline{Hom}^{geom}(T,T')$ (it is defined to be the full sub-Segal category of $\mathbb{R}\underline{Hom}(T',T)$ consisting of exact morphisms which admit a right adjoint). For an object $p : T \longrightarrow T'$ in $\mathbb{R}\underline{Hom}^{geom}(T,T')$ we will denote by p^* the corresponding object in $\mathbb{R}\underline{Hom}(T',T)$ and by $p_* \in \mathbb{R}\underline{Hom}(T,T')$ its right adjoint.

In order to state the theorem, let us call a stack $F \in T$ in a Segal topos T *constant* if it is in the essential image of π^*, the inverse image of the unique morphism of Segal topoi $\pi : T \longrightarrow Top$. A stack F will be called *weakly locally constant* [4] if there exists an epimorphism $X \longrightarrow *$ in T such that $F \times X \longrightarrow X$ is a constant stack in the Segal topos T/X. When $T = St(X)$ is the topos of stacks on a topological space X, then a stack $F \in St(X)$ is weakly locally constant if and only if it is locally equivalent to a constant simplicial presheaves.

Theorem 3.4 *(Toën-Vezzosi, [To-Ve1, Thm. 5.2.1])*

1. *For any CW complex X let $St(X)$ be the Segal category of stacks on X. The full sub-Segal category $Loc(X)$ of $St(X)$, consisting of locally constant stacks on X is a t-complete Segal topos.*

2. *For two CW complexes X and Y, the Segal category of geometric morphisms $\mathbb{R}\underline{Hom}^{geom}(Loc(X), Loc(Y))$ is a Segal groupoid. Furthermore, there is a nat-*

[4] The expression *locally constant* will be reserved for a stronger notion that will be introduced later in the text.

ural equivalence of simplicial sets

$$Map(X,Y) \simeq |\mathbb{R}\underline{Hom}^{geom}(Loc(X), Loc(Y))|,$$

where the right hand side is the nerve of the Segal groupoid $\mathbb{R}\underline{Hom}^{geom}(Loc(X), Loc(Y))$.

3. *The morphism of* 2*-Segal categories*

$$Top \longrightarrow \{Segal\ Topoi\},$$

sending X *to the Segal topos* $Loc(X)$ *is fully faithful.*

The theorem has the following important consequence. It is a Segal category version of *hypothèse inspiratrice* of [Gr2], stating that the homotopy category of spaces does not have any non-trivial auto-equivalences.

Corollary 3.5 *Let* $\mathbb{R}\underline{Aut}(A)$ *be the full sub-Segal category of* $\mathbb{R}\underline{Hom}(A,A)$ *consisting of equivalences. Then, one has* $\mathbb{R}\underline{Aut}(Top) = simeq*$.

Based on the previous theorem, for any Segal topos T we define its *homotopy shape* to be the morphism of Segal categories

$$H_T : Top \longrightarrow Top,$$

sending X to $|\mathbb{R}\underline{Hom}^{geom}(T, loc(X))|$. The key observation, base on theorem 3.4, is that if $T = Loc(Y)$ for a CW complex Y then H_T is co-represented (in the sense of Segal categories) by the homotopy type of Y. In the general situation the functor H_T is only co-representable by a pro-object in the Segal category Top. Precisely, one can prove the following pro-representability result. Its proof essentially relies on the fact that the homotopy shape H_T is a left exact morphism of Segal categories. Because of some technical difficulties we will assume that T is a t-complete Segal topos, but I expect the proposition to be correct in general (this complication is related with the problem of defining a reasonable notion of hyper-topologies as explained after theorem 3.3).

Proposition 3.6 *(Toën-Vezzosi, see [To-Ve1]) Let* T *be a* t*-complete Segal topos and* H_T *its homotopy shape as defined above. There exists a left filtered Segal category* A *(e.g. which possesses finite limits) and a morphism* $K_T : A \longrightarrow Top$*, which co-represents the morphism* H_T*. In other words, the following two endomorphisms of the Segal category* Top

$$X \mapsto H_T(X) \qquad X \mapsto Hocolim_{a \in A^{op}} Map(K_T(a), X)$$

are equivalent.

In the last proposition there is a universe issue that is not mentioned. In fact, if T is a $\mathbb{U}$-Segal topos (i.e. the Segal category T in definition 3.1 can be chosen to be $\mathbb{U}$-small) then the Segal category A can be chosen to be $\mathbb{U}$-small.

Definition 3.7 *The morphism of Segal categories $K_T : A \longrightarrow Top$ of proposition 3.6 is called the* pro-homotopy type of the Segal topos T.

The pro-homotopy type K_T of a Segal topos T is not a pro-object in the category of simplicial set. Indeed, it is only a pro-object in the sense of Segal categories (i.e. the category of indices A is Segal category), but it seems however that this notion is not strictly more general than the usual notion of pro-simplicial sets (see [Lu1, App. B]).

The Segal category A of definition 3.7 is left filtered, and it follows that so is its homotopy category $Ho(A)$. In particular, the morphism $K_T : A \longrightarrow Top$ gives rise to a pro-object $Ho(K_T) : Ho(A) \longrightarrow Ho(Top)$, which is a pro-object in the homotopy category of spaces. When the Segal topos T is in fact the Segal category of stacks over a locally connected Grothendieck site C, I suspect that the pro-object $Ho(K_T) : Ho(A) \longrightarrow Ho(Top)$ is isomorphic to the Artin-Mazur's pro-homotopy type of C as defined in [Ar-Ma] (at least after some $\sharp$-construction). The pro-homotopy type K_T of definition 3.7 is therefore a refinement of Artin-Mazur's construction, already for the case of locally connected Grothendieck sites.

The fundamental property of the pro-homotopy type K_T of a Segal topos T is the following. In order to state it let us first start by some general notions. We fix a t-complete Segal topos T and we simply let K be its pro-homotopy type of definition 3.7.

- Let $\underline{Top}$ be the constant Segal stack over T associated with the Segal category Top (the theory of Segal stacks of [Hi-Si] generalizes in an obvious way to Segal stacks over Segal topoi). The *Segal category of locally constant stacks on T* is defined to be
$$Loc(T) := \mathbb{R}\underline{Hom}(*, \underline{Top}),$$
where the right hand side is the Segal category of (derived) morphisms of Segal stacks over T. I warn the reader that the natural morphism $Loc(T) \longrightarrow T$, given by descent theory, is not fully faithful in general. The Segal category $Loc(T)$ is therefore not a full sub-Segal category of T in general, and its objects consist of objects of T endowed with certain additional structures. Also, the notion of locally constant stack on T is clearly different from the notion of weakly locally constant stacks used before. Not even every weakly locally constant stack in T lies in the essential image of the morphism $Loc(T) \longrightarrow T$. However, when T is locally contractible (in particular for $T = St(X)$ for a CW complex X), then $Loc(T) \longrightarrow T$ is fully faithful and its image consists precisely of all weakly locally constant stacks.

- Let $K : A \longrightarrow Top$ be the pro-homotopy type of the Segal topos T. Composing with the morphism $X \mapsto Loc(X)$ one gets an ind-object in the 2-Segal category of Segal categories

$$A^{op} \xrightarrow{K} Top^{op} \xrightarrow{Loc(-)} \underline{SeCat}.$$

The colimit of this morphism (in the 2-Segal category $\underline{SeCat}$) is denoted by

$$Loc(K) := Colim_{a \in A^{op}} Loc(K_a) \in \underline{SeCat},$$

and is called the *Segal category of locally constant stacks over* K.

The above unpublished theorem is the universal property of the pro-homotopy type K of the Segal topos T.

Theorem 3.8 *With the above notations, there exists a natural equivalence of Segal categories*

$$Loc(T) \simeq Loc(K).$$

The theorem 3.8 gives a universal property of the pro-homotopy type associated to a Segal topos, which is a generalization of the well known universal property of the fundamental groupoid of a locally connected topos. When T is the Segal topos of stacks over a locally connected Grothendieck site, this universal property were not known (and is probably uneasy or even impossible to state) for Artin-Mazur's homotopy type, as the definition of $Loc(K)$ really uses the fact that K is a pro-object in Top and not only in its homotopy category.

4 Homotopical algebraic geometry

The main references for this section are [To-Ve1, To-Ve3, To-Ve4, To-Ve5, To-Ve6].

Developing homotopical algebraic geometry ("HAG" for short) is a project we started together with G. Vezzosi during the fall 2000, and which is still in progress. The main goal of HAG is to provide a mathematical setting in which one can talk about schemes in a context were affine objects are modelled by homotopy-ring-like objects (e.g. commutative differential graded algebras, commutative ring spectra, symmetric monoidal categories ...). We already know that algebraic geometry possesses generalization to a relative setting for which affine objects are modelled by commutative rings in a general ringed topos or in a Tannakian category (see [De1, Hak]). The new feature appearing in HAG is the fact that the category of models for affine objects comes with a non-trivial homotopy theory (e.g. a model category structure) that have to be taken into account. Clearly, *homotopical algebraic geometry is to algebraic geometry what homotopical mathematics are to mathematics.*

The original motivations for starting such a project were various. As an indication for the reader, I remember below three of them (see also [To-Ve4, To-Ve5] for more details).

- *Tannakian duality:* Form the algebraic point of view the Tannakian dual of a (neutral) Tannakian category (over a field k) is a commutative Hopf k-algebra, which from the geometrical point of view corresponds to an affine group scheme over k. In the (conjectural) Tannakian formalism for Segal categories (see §5) the base monoidal category of k-vector spaces is replaced by the monoidal Segal category of complexes of k-vector spaces. The Tannakian dual of a (neutral) Tannakian Segal category is therefore expected to be *a commutative Hopf algebra in the monoidal Segal category of complexes* (i.e. some kind of commutative dg-Hopf k-algebra), and geometrically one would like to consider this Hopf algebra as an *affine group scheme over the Segal category of complexes of k-vector spaces*. My original interests in this notion of Tannakian Segal category were based on the observation that several interesting derived categories appearing in algebraic geometry (e.g. derived category of perfect complexes of local systems or l-adic sheaves, of perfect complexes with flat connections, of perfect complexes of F-isocrystals ...) are in fact the homotopy categories of natural Tannakian Segal categories, whose dual can be considered as certain homotopy types (in the same way as the dual of a neutral Tannakian category is considered as a fundamental group, or more generally as a 1-truncated homotopy type). This point of view, which can be avoided to actually construct these homotopy types (see §6), seems to me very powerful for the study of homotopy types in algebraic geometry (see §6.4).

- *Derived algebraic geometry:* There are essentially two kind of general constructions in the category of schemes, colimits (e.g. quotients) and limits (e.g. fibered products). These two constructions are not *exact* in some sense and according to a very general philosophy they should therefore be *derived*. Stacks and algebraic stacks theory has been introduced in order to be able to make *derived quotients*. More generally, higher stacks provide a theory in which one can do arbitrary *derived colimits*. On the other side, the notion of *dg-schemes* (see e.g. [Cio-Kap1, Cio-Kap2]) have been introduced in order to be able to do *derived fibered products*, and more generally *derived limits*. However, this approach have encountered two major problems, already identified in [Cio-Kap2, 0.3].

  1. The definition of dg-schemes and dg-stacks seems too rigid for certain purposes. By definition, a dg-scheme is a space obtained by *gluing commutative differential graded algebras for the Zariski topology*. It seems however that certain constructions really require a weaker notion of gluing, as for example *gluing differential graded algebras up to quasi-isomorphisms* (and a weaker topology).

2. The notion of dg-schemes is not very well suited with respect to the functorial point of view, as representable functors would have to be defined on the derived category of dg-schemes (i.e. the category obtained by formally inverting quasi-isomorphisms of dg-schemes), which seems difficult to describe and to work with. As a consequence, the derived moduli spaces constructed in [Kap1, Cio-Kap1, Cio-Kap2] do not arise as solution to natural *derived moduli problems*, and are constructed in a rather ad-hoc way.

The main idea to solve these two problems was to interpret dg-schemes as *schemes over the category of complexes*. Therefore, it appeared to us that the theory of dg-schemes should be only an approximation of what algebraic geometry over the category of complexes is. Such a theory actually did provide to a us a context in which wa have been able to construct many new derived moduli spaces that were not constructed (and probably could not be constructed) as dg-schemes or dg-stacks.

- *Moduli spaces of multiplicative structures:* Fixing a finite dimensional vector space V, one can define the classifying stack of algebra structures on V, Alg_V. This is an algebraic stack in the sense of Artin, which is a solution to the classification problem of algebra structures on V.

  In algebraic topology, the classification problem of ring structure on a given spectra X appears naturally and in several interesting contexts (see [Goe-Hop, Laz]). However, this classification problem has, until now, only been solved in a rather crude way by using classifying spaces as in [Re1]. These classifying spaces are homotopy types, and therefore are discrete invariants (in our previous example they would correspond to the space of global sections of the stack Alg_V alone, thus it looks like the underlying set of points of an algebraic variety). It seems therefore very natural to look for additional algebraic structures on the classifying space of multiplicative structures on a given spectra X, reflecting some global geometry. The main idea is be to define a *classifying stack* Alg_X, of algebra structures on X, in a very similar manner than the stack Alg_V is defined. Our point of view was that the stack Alg_X only exist in a reasonable sense as a *stack over the category of spectra*, and therefore belongs to *algebraic geometry over spectra*. As we will see below, HAG actually provide a context in which this construction, and several others, makes sense.

In the following paragraph I will present the general theory, as well as some examples of constructions of moduli spaces in this new context. Details can be found in [To-Ve3, To-Ve4, To-Ve5].

4.1 HAG: The general theory

The general formalism was inspired to us by the work of C. Simpson around the notion of n-geometric stacks, as exposed in [Si6]. The only new difficulty here is to take into account correctly the homotopy theory of the category of affine objects, which is possible thanks to our work on Segal sites, stacks and Segal topoi presented before.

The starting point is a symmetric monoidal model category M which will be the base category of the theory (se also remark 4.2). One considers the category $Comm(M)$, of E_∞-algebras in M, which is very often a model category for which equivalences are simply defined on the underlying objects in M (I assume that it is for simplicity, see [Sp] for details). The model category of *affine schemes over* M is defined to be the opposite model category $Aff_M := (Comm(M))^{op}$, and its Dwyer-Kan localization $LAff_M$ is called the *Segal category of affine schemes over* M. We assume that one is given a topology τ on the Segal category $LAff_M$ (i.e. a Grothendieck topology on the homotopy category $Ho(Aff_M)$). Of course, finding interesting topologies on $LAff_M$ very much depends on the context and could be sometimes not so easy (examples will be given in the next paragraphes). The Segal category $LAff_M$ together with the topology τ is a Segal site in the sense of Def. 3.2, and we can therefore apply the general theory of Segal topoi and stacks in order to produce the Segal topos of stacks for the topology τ, denoted by $LAff_M^{\sim,\tau}$. The Segal topos $LAff_M^{\sim,\tau}$ is the fundamental object in order to *do algebraic geometry over* M, and plays exactly the same role as the category of sheaves on the big site of schemes in classical algebraic geometry. As explained in [To-Ve3, §4], there is a natural model category of stacks $Aff_M^{\sim,\tau}$ whose simplicial localization is equivalent to $LAff_M^{\sim,\tau}$. In practice the existence of the model category $Aff_M^{\sim,\tau}$ is very helpful as it allows to work within a model category rather than in a Segal category and reduces statements for Segal category theory to usual category theory. Because of this, very often, objects in $LAff_M^{\sim,\tau}$ will be implicitly considered as objects of the model category $Aff_M^{\sim,\tau}$. This has the advantage that stacks will be represented by actual functors (in the usual sense)

$$F : Comm(M) \longrightarrow SSet,$$

from the category of E_∞-algebras in M to the category of simplicial sets, and which satisfies the following two conditions:

- The functor F preserves equivalences.
- The functor F has the descent property with respect to hyper-coverings (see [To-Ve3, §4.4] for details).

As for any Segal category, one has a Yoneda embedding morphism

$$h : LAff_M \hookrightarrow L\hat{A}ff_M.$$

For an E_∞-algebra $R \in Comm(M)$, its image by the morphism h is usually denoted by $\mathbb{R}Spec\, R$. We will assume that the topology is *sub-canonical*, or equivalently that the Yoneda embedding factors through the sub-Segal category of stacks

$$h : LAff_M \hookrightarrow LAff_M^{\sim,\tau} \subset \hat{LAff}_M.$$

Objects of $LAff_M^{\sim,\tau}$ lying in the essential image of h will be called *representable stacks*, and will play the role of affine schemes in the whole theory. Directly related to this, one defines the notion of representable morphisms between stacks. A morphism $f : F \longrightarrow G$ in $LAff_M^{\sim,\tau}$ is *representable* if for any representable stack H, and any morphism $H \to G$, the fiber product (in the Segal category $LAff_M^{\sim,\tau}$, or equivalently the homotopy fibered product in the model category $Aff_M^{\sim,\tau}$) $F \times_G H$ is a representable stack.

Let us now assume that one has a notion P of morphisms in the Segal category $LAff_M$. We suppose that morphisms in P are stable by compositions and pull-backs. We also assume some compatibilities between morphisms in P and the topology τ that I will not explicitly state here. For example, we assume that a morphism which locally (for the topology τ) is in P lies itself in P. I will not describe all of these conditions and the reader is advised to make the comparison with the case were τ is the usual étale topology on affine schemes and P consists of all smooth or flat morphisms. As usual, the notion of being in P can be extended from morphisms in $LAff_M$ to representable morphisms in $LAff_M^{\sim,\tau}$. Precisely, a representable morphism $f : F \longrightarrow G$ is in P if for any representable stack H the induced morphism $F \times_G H \longrightarrow H$ is in P (this makes sense since both stacks are representable and therefore this last morphism can be considered as a morphism in $LAff_M$. We use here the fact that the topology τ is sub-canonical.).

The main definition of HAG is the following. It is the obvious extension of the notion of algebraic stacks to our general setting.

Definition 4.1 *A stack $F \in LAff_M^{\sim,\tau}$ is called* P-geometric *if it satisfies the following conditions.*

1. *The diagonal morphism $\Delta : F \longrightarrow F \times F$ is representable.*
2. *There exists a family of representable stacks $\{U_i\}$, and an epimorphism*

$$p : \coprod_i U_i \longrightarrow F$$

such that each $p_i : U_i \longrightarrow F$ is in P (note that each p_i is automatically representable by condition (1)).

Without going into details (it would be too long) let me mention that the above definition can be iterated in order to define the notion of *$n-P$-geometric stacks* for an integer n. I refer to [Si6] and [To-Ve5, §3.3] for some details on this notion. The

very general definition will also appear in [To-Ve6].

The general theory of geometric stacks can then be pursued in parallel with the usual theory of schemes and of algebraic stacks. One can for example define notions of sheaves of modules, cohomology, K-theory .... Almost all general notions available for algebraic stacks are also available for geometric stacks in the sense of definition 4.1. Of course, all of the general theory is very formal and non-trivial mathematical statements and constructions really start when one specializes the base model category M. In the following I will gives three example, corresponding to the case were M is the model category of negatively graded complexes, of unbounded complexes and of spectra. Most of the proofs will appear in [To-Ve6].

Remark 4.2 The starting point in the theory presented above were a base symmetric monoidal model category M. There also exists a *model category free* approach, for which the starting point is a symmetric monoidal Segal category S (see Def. 5.1 for definitions). The two point of views can be compared using the fact that LM is a symmetric monoidal Segal category when M is a symmetric monoidal model category. This is of course a much more general theory but which requires a lot of technology (essentially linear algebra in monoidal Segal categories, as algebras, modules, ...) in order to be done. Ultimately, working entirely in the Segal setting and not refereing to model categories would have great advantages, but it seems to that Segal category theory is not well developed enough in order for such an approach to be reasonable.

4.2 DAG: Derived algebraic geometry

For introduction to derived algebraic geometry I refer to [To-Ve4], in which we have tried to exposed the general motivations, the main philosophy and part of the history of the subject.

In this part we specialize M to be the model category $C^-(k)$ of negatively graded (with increasing differential) complexes of k-modules, for k a commutative ring. It is a symmetric monoidal model category for the projective model structure of [Ho, §2.3], and one can therefore apply the general constructions presented in the last part. The theory presented below gives an alternative to the theory of dg-schemes of [Cio-Kap1, Cio-Kap2], having the advantage of providing a functorial point of view which seems difficult to deal with inside the theory of dg-schemes. It is also general enough in order to deal with objects as *dg-stacks* and *higher dg-stacks* for which the approach of dg-schemes does not seem to be always well suited. As a consequence, we are able to consider moduli problems that does not seem to have been constructed before, and in particular we construct global counter parts of the formal moduli spaces considered in [Kon, Kon-Soi].

First of all, one defines an étale topology on the Segal category $LAff_{C^-(k)}$ of affine schemes over $C^-(k)$ in the following way (recall that the category $Aff_{C^-(k)}$ has been defined as the opposite of the category of E_∞-algebras in the model category $C^-(k)$, or in other words of non-positively graded E_∞-algebras over k). A morphism $f : A \longrightarrow B$ of E_∞-algebras (over k) is *étale* (resp. *strongly smooth*, resp. *strongly flat*) if it satisfies the following three conditions.

- The morphism $A \longrightarrow B$ is finitely presented in the sense of Segal categories[5]. Equivalently, this means that B is equivalent to a retract of a finite cell A-algebra in the sense of [EKMM] for example.
- The induced morphism $H^0(A) \longrightarrow H^0(B)$ is an étale (resp. smooth, resp. flat) morphism of rings.
- The natural morphism $H^*(A) \otimes_{H^0(A)} H^0(B) \longrightarrow H^*(B)$ is an isomorphism.

Now, a family of morphisms of E_∞-algebras $\{A \to B_i\}_{i\in I}$ is an *étale covering* if each morphism $A \to B_i$ is étale, and if the family $\{Spec\, H^0(B_i) \to Spec\, H^0(A)\}_{i\in I}$ is an étale covering of affine schemes. This defines an étale topology on the Segal category $LAff_{C^-(k)}$, and therefore a Segal topos of stacks $LAff^{\sim,et}_{C^-(k)}$. Furthermore, by taking P to be the set of strongly smooth morphisms in the sense above, one obtains the notion of *strongly geometric stacks in* $LAff^{\sim,et}_{C^-(k)}$ (these are called *strongly geometric D-stacks* in [To-Ve4]).

I will not recall here all of the properties of stacks and strongly geometric stacks. Let me however mention two important facts.

- Let $LAff^{\sim,et}_k$ be the Segal category of stacks on the Grothendieck site of affine k-schemes with the étale topology. The natural inclusion functor from the category of commutative k-algebras inside the category of E_∞-algebras induces a fully faithful morphism of Segal categories

$$i : LAff^{\sim,et}_k \longrightarrow LAff^{\sim,et}_{C^-(k)},$$

  which admits a right adjoint

$$h^0 : LAff^{\sim,et}_{C^-(k)} \longrightarrow LAff^{\sim,et}_k.$$

  However, the morphism i is not left exact (as push-outs of commutative k-algebras do not coincide in general with homotopy push-outs of E_∞-algebras), and therefore the adjoint pair (i, h^0) does not define a geometric morphism of Segal topoi. The fact that i does not commutes with fiber products is one of the key feature of derived algebraic geometry: *taking fiber products of schemes might not be a scheme anymore.*

[5]Recall that an object x in a Segal category A is finitely presented if the morphism $A_{(x,-)} : A \longrightarrow Top$ commutes (up to equivalence) with filtered colimits. A morphism $x \to y$ in A is finitely presented if y is a finitely presented as an object in the coma Segal category x/A.

For any stack $F \in LAff^{\sim,et}_{C^-(k)}$, the stack $h^0(F)$ is called the *truncation of* F. When F is a strongly geometric stack, then $h^0(F)$ is an algebraic stack in the sense of Artin, and the natural morphism $ih^0(F) \longrightarrow F$ is a *closed embedding*. This is the general picture of a *classical moduli space* $h^0(F)$ sitting inside its *derived version* F. The closed embedding $ih^0(F) \hookrightarrow F$ behave very much as a formal thickening , and the geometry of the two spaces are essentially the same whereas their structural sheaves can be very different.

- For a stack $F \in LAff^{\sim,et}_{C^-(k)}$ one can define its tangent stack TF to be the stack of morphisms from $Spec\, k[\epsilon]$ to F. It comes with a natural projection $TF \longrightarrow F$, whose fiber over a global point x of F is defined to be the *tangent space of* F *at* x. When the stack F is strongly geometric, its tangent space T_xF at a point x is a *linear stack*. This means that T_xF correspond to a complex of k-modules, which is furthermore concentrated in degree $[-1, \infty[$.

  One should be careful that for a stack $F \in LAff^{\sim,et}_{k}$, and in particular for a scheme, one does not have $iTF \simeq T(iF)$. Indeed, if X is a scheme then iTX is the global space of the tangent sheaf on X (i.e. $iTX = Spec\, Symm(\Omega^1_X)$), whereas $T(iX)$ is the global space of Illusie's tangent complex, $T(iX) = \mathbb{R}Spec\, Symm(\mathbb{L}_X)$.

We are now ready to define various derived moduli functors. As an examples I will describe the stack $\mathbb{R}Ass$, classifying the associative algebra structures. Other examples are described in [To-Ve4].

Let V be a projective k-module of finite type. For an E_∞-algebra A, one considers the category $Alg_V(A)$ whose objects are associative A-algebras whose underlying A-module is, locally for the étale topology on A, equivalent to $A \otimes_k V$, and whose morphisms are equivalences of A-algebras. The simplicial set $\mathbb{R}Ass_V(A)$ is defined to be the geometric realization of the category $Alg_V(A)$. For a morphism of E_∞-algebras $A \longrightarrow A'$ one has a base change morphism $\mathbb{R}Ass_V(A) \longrightarrow \mathbb{R}Ass_V(A')$ sending a A-algebra B to the A'-algebra $B \otimes^{\mathbb{L}}_A A'$. This defines a functor

$$\mathbb{R}Ass_V : Comm(C^-(k)) \longrightarrow SSet,$$

satisfying the conditions for being a stack, and therefore and object in $LAff^{\sim,et}_{C^-(k)}$.

Theorem 4.3 *(Toën-Vezzosi) The stack $\mathbb{R}Ass_V$ is strongly geometric. The tangent space of $\mathbb{R}Ass_V$ at a global point $B \in \mathbb{R}Ass_V(k)$, corresponding to an associative algebra structure on V, is naturally identified with the shifted complex of derived derivation $\mathbb{R}Der(B,B)[1]$.*

The strongly geometric stack $\mathbb{R}Ass_V$ has a truncation $h^0(\mathbb{R}Ass_V)$, which is the usual Artin stack Ass_V of associative algebra structures on the k-module V. The stack Ass_V sits as a closed sub-stack inside $\mathbb{R}Ass_V$. This is a very general situation, for any strongly geometric stack F the truncation $h^0(F)$ lives in F as a closed

sub-stack. Furthermore, the usual tangent space of Ass_V at the point B can be identified with the truncation $\tau_{\leq 0}\mathbb{R}Der(B,B)[1]$, explaining, at least infinitesimally, the additional information contained in the derived version $\mathbb{R}Ass_V$. The main advantage of the derived version $\mathbb{R}Ass_V$ is that its tangent spaces can be explicitly identified as the complexes of (derived) derivations, whereas the cotangent complex of the truncation Ass_V remains a mystery (though is tangent space is understood). This is, as far as I understand, one of the main feature of derived moduli spaces; their tangent complexes have very natural geometric descriptions which allows one to describe the local nature of the moduli space, whereas the deformation theory of the truncated version is very much unnatural and difficult to deal with. This observation is apparently the main point of the *derived deformation theory program* of P. Deligne, V. Drinfeld and M. Kontsevich.

Without going into details, let me mention that one can define many other derived moduli problems (e.g. local systems on a space, linear category structures, vector bundles on a schemes ...) and prove that their are strongly geometric stacks in many cases (see [To-Ve4, §5]). The very general point of view of our approach has also allowed us to define certain moduli stacks that have never been considered before (as for example the 2-geometric stack of linear categories described in [To-Ve4, §5.3]).

Clearly, derived algebraic geometry requires several basic and fundamental results in order to be as easy to manipulate as usual algebraic geometry. My feeling is that derived geometric stacks behave in a quite similar fashion than schemes and several (if not all) of the results of the EGA's could be generalized to the derived setting, as for example cohomology of projective spaces, cohomology and base changes, the local structures of étale and smooth morphisms, Grothendieck's existence theorem .... Quite recently (see [Lu2]), J. Lurie has announced a version of Artin's representability theorem in the context of derived algebraic geometry, which is definitely going to be extremely useful for the construction of more examples of geometric derived stacks.

4.3 UDAG: Unbounded derived algebraic geometry

In this part I want to present a natural extension of derived algebraic geometry of the last paragraph, in which the base model category of bounded above complexes $C^-(k)$ is replaced with the model category of unbounded complexes $C(k)$. This extension allows one to define and study more moduli problems that do not naturally enter the setting of the last paragraph, as for example the generalization of the stack $\mathbb{R}Ass_V$ when V is a complex of k-modules rather than just a k-module (i.e. the classifying stack of dga structures on a complex V). Another natural example is the stack of fiber functors of a k-tensor Segal category mentioned in the motivations of this chapter, and which will be investigated in §5.

Let now M be the model category of unbounded complexes $C(k)$ with the projec-

tive model structure, which is a symmetric monoidal model category (see [Ho, §2.3]). The category $Aff_{C(k)}$ is now the opposite category of unbounded E_∞-algebras over k. One defines the étale topology on $LAff_{C(k)}$ exactly the same way as for the bounded case (note that the definition given in the last paragraph is also valid for unbounded E_∞-algebras). Associated to this topology is the Segal category of stacks $LAff^{\sim,et}_{C(k)}$.

The inclusion functor $i : C^-(k) \longrightarrow C(k)$ induces an adjunction on the Segal categories of stacks

$$i_! : LAff^{\sim,et}_{C^-(k)} \longrightarrow LAff^{\sim,et}_{C(k)} \qquad LAff^{\sim,et}_{C^-(k)} \longleftarrow LAff^{\sim,et}_{C(k)} : i^*,$$

and the morphism $i_!$ is in fact fully faithful. The notion of stacks over unbounded E_∞-algebras is therefore a generalization of the notion of stacks over bounded above E_∞-algebras. In particular, there exists natural fully faithful morphisms

$$LAff^{\sim,et}_k \longrightarrow LAff^{\sim,et}_{C^-(k)} \longrightarrow LAff^{\sim,et}_{C(k)},$$

each of them having right adjoints. This allows one to consider schemes, algebraic stacks and strongly geometric stacks as defined in the last paragraph as objects in $LAff^{\sim,et}_{C(k)}$.

It turns out that the notion of strong smoothness used in the last paragraph is not going to be general enough for our new examples. Indeed, we will be lead to take quotient by representable group stacks in $LAff^{\sim,et}_{C(k)}$, and a representable stack which is strongly smooth over $Spec\, k$ is automatically a group scheme in the usual sense. We therefore extend the notion of strongly smooth morphisms in the following way (this is what is called *standard smoothness* in [To-Ve4, §4.4]). It is based on the usual characterization of smooth morphisms of schemes as the morphisms which étale locally looks like the projection of a vector bundle.

A morphism of E_∞-algebra $f : A \longrightarrow B$ is *smooth* if there exists a commutative square (up to homotopy)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A' & \xrightarrow[f']{} & B', \end{array}$$

such that

- v is an étale covering.
- f' is an étale morphism.
- The A-algebra A' is equivalent to $A \otimes^{\mathbb{L}}_k L(E)$, where E is a perfect complex of k-modules and $L(E)$ is the free E_∞-algebra over E.

By taking P to be the set of smooth morphisms as defined above one obtains by definition Def. 4.1 the notion of a *geometric stack* in $LAff^{\sim,et}_{C(k)}$. One should be careful however that an object $F \in LAff^{\sim,et}_{C^-(k)}$ can be geometric as an object in $LAff^{\sim,et}_{C(k)}$ without being strongly geometric. To clarify, one has natural inclusions

$$\{Schemes\} \hookrightarrow \{Alg.\ stacks\} \hookrightarrow \{Strong.\ geom.\ stacks\} \hookrightarrow \{Geom.\ stacks\} \hookrightarrow LAff^{\sim,et}_{C(k)}$$

We can now generalize the construction of the stack $\mathbb{R}Ass$ in the following way. Let V be a perfect complex of k-modules. For a E_∞-algebra A, one considers the category $Alg_V(A)$ whose objects are associative A-algebras whose underlying A-module is, locally for the étale topology on A, equivalent to $A \otimes^{\mathbb{L}}_k V$, and whose morphisms are equivalences of A-algebras. The simplicial set $\mathbb{R}Ass_V(A)$ is defined to be the geometric realization of the category $Alg_V(A)$. For a morphism of E_∞-algebras $A \longrightarrow A'$ one has a base change morphism $\mathbb{R}Ass_V(A) \longrightarrow \mathbb{R}Ass_V(A')$ sending a A-algebra B to the A'-algebra $B \otimes^{\mathbb{L}}_A A'$. This defines a functor

$$\mathbb{R}Ass_V : Comm(C(k)) \longrightarrow SSet,$$

satisfying the stack conditions, and therefore and object in $LAff^{\sim,et}_{C(k)}$.

Theorem 4.4 *(Toën-Vezzosi) The stack $\mathbb{R}Ass_V$ is geometric. The tangent space of $\mathbb{R}Ass_V$ at a global point B, corresponding to a dga structure on V, is naturally identified with the shifted complex of derived derivation $\mathbb{R}Der(B,B)[1]$.*

Remark 4.5 The geometric stack $\mathbb{R}Ass_V$ also has a truncation $h^0\mathbb{R}Ass_V =: Ass_V$, which is a stack in $LAff^{\sim,et}_k$. This stack is not an algebraic stack in the sense of Artin anymore, and is actually a non-truncated stack in general. However, one can show that Ass_V is equivalent to a quotient stack $[F/G]$, where F is an affine stack in the sense of [To2], and G is an affine group stack acting on F. This shows that the stack Ass_V still has some kind of algebraic nature (see [To2, 4.2.1]).

It is sometimes useful to consider an associative k-algebra as a k-linear category with a unique object. In the same way, a dga can be considered as a dg-category (over k) with a unique object. It would be too long to explain in details, but one can define a stack $\widetilde{RCat_V}$ which classifies structures of dg-categories having a unique object and V as complex of endomorphisms (see [To-Ve4, §5.3]). This stack can not be geometric, but one can show it is a 2-*geometric stack*. Considering a dga as a dg-category gives a morphism of stacks $\mathbb{R}Ass_V \longrightarrow \widetilde{RCat_V}$, which is a smooth fibration. Indeed, the fiber of this morphism over the image of dga B can be identified with $K(Gl_1(B),1)$, the classifying stack of the group stack of invertible elements in the dga B (see [To-Ve4, §4.3] for more details). Therefore, the above projection morphism induces an exact triangle on the level of tangent spaces, that can be written as

$$B[1] \longrightarrow \mathbb{R}Der_k(B,B)[1] \longrightarrow C^+_k(B,B)[2], \xrightarrow{+1}$$

where $C_k^+(B,B)$ is the complex of Hochschild cohomology of the dga B. This is the fundamental triangle appearing in [Kon, p. 59]. The fibration sequence (in the Segal category $LAff^{\sim,et}_{C(k)}$)

$$K(Gl_1(B),1) \longrightarrow \mathbb{R}Ass_V \longrightarrow \widetilde{\mathbb{R}Cat_V}$$

gives a geometric explanation of the triangle [Kon, p. 59]. The object $\widetilde{\mathbb{R}Cat_V}$ seems to right think to consider in order to explain deformation theory of associative dg-algebras and dg-categories in the spirit of [Kon-Soi, Soi].

Finally, the 2-geometric stack of dg-categories $\widetilde{\mathbb{R}Cat}$ can have some interesting applications in the context of mirror symmetry which some people think about as some involution of the stack $\widetilde{\mathbb{R}Cat}$ suitably modified (one needs to consider dg-categories which are *non-commutative Calabi-Yau manifolds* in the sense of [Soi]).

4.4 BNAG: Brave new algebraic geometry

The expression *brave new algebra* was introduced by F. Waldhausen. As far as I understand it reflects the fact that ring spectra behave very much the same way as ordinary rings, and that many basic constructions in the context of rings (tensor product, Tor, Ext, algebra of matrices, K-theory, Hochschild cohomology, ...) has reasonable analogs in the context of ring spectra (see [Vo] for an introduction on the subject). We like to use the expression *brave new algebraic geometry* for the special case of HAG in which the base model category M is chosen to be the category of spectra. One can say that *brave new algebraic geometry is to algebraic geometry what brave new algebra is to algebra*.

The setting of brave new algebraic geometry seems to be a very well suited setting for many recent works in stable homotopy theory, as elliptic cohomology (see [Goe]), moduli spaces of multiplicative structures (see [Goe-Hop, Laz]) .... In this part I will present very basic constructions, as we did not investigate yet more serious examples (see however [To-Ve7] for some more details).

We let M to be the model category Sp^Σ, of symmetric spectra as defined in [Ho-Sh-Sm]. It is a symmetric monoidal model category for the smash product. We endow the Segal category $LAff_{Sp^\Sigma}$ (the opposite Segal category of E_∞-ring spectra[6]) with an étale topology defined in the following way. A morphism of E_∞-ring spectra $A \longrightarrow B$ is *étale* if it satisfies the following properties.

- The A-algebra is finitely presented (in the sense of Segal categories).
- The cotangent spectra $\mathbb{L}_{B/A}$ is acyclic.

[6] An very convenient model for the homotopy theory of E_∞-ring spectra is exposed in [Shi].

A finite family of morphisms of E_∞-ring spectra $\{A \longrightarrow B_i\}_{i\in I}$ is an *étale covering* if each morphisms $A \longrightarrow B_i$ is étale, and if the family of base change functors of the category of modules

$$\begin{array}{ccc} Ho(Mod(A)) & \longrightarrow & Ho(Mod(B_i)) \\ M & \mapsto & M \wedge^{\mathbb{L}}_A B_i \end{array}$$

is a conservative family (i.e. a A-module M is acyclic if and only if each B_i-modules $M \wedge^{\mathbb{L}}_A B_i$ is acyclic).

This defines an étale topology on $LAff_{Sp^\Sigma}$, and therefore one has an associated Segal category of stacks $LAff^{\sim,et}_{Sp^\Sigma}$. The existence of this Segal category already allows to make sense of certain constructions that have been informally considered by J. Rognes (see [Ro]). First of all, for a E_∞-ring spectra R, one can consider its small étale site $(\mathbb{R}Spec\, R)_{et}$, as well as its associated Segal topos. By the general construction 3.7 one extract from this Segal topos a pro-homotopy type, called the *étale homotopy type of the E_∞-ring spectra R*. Following the same kind of ideas, one can define étale K-theory of an E_∞-ring spectra, $K_{et}(R)$, which comes naturally with a localization morphism $K(R) \longrightarrow K_{et}(R)$ (see [To-Ve3]). One could then try to follow the same guide line as for usual rings, like defining a Thomason's style topological K-theory, state a Quillen-Lichtenbaum conjecture .... As far as I understood from [Ro] and talks by J. Rognes, one of the main goal of the whole machinery would be to compute, at least partially, the algebraic K-theory of the sphere spectra which appears naturally in geometric topology (the sphere spectra $\mathbb{S}$ is a direct factor of the algebraic K-theory of the point space $*$).

To finish this part, let me mention that one can also define a notion of smooth morphisms of E_∞-ring spectra, in the exact same way that we have defined it for E_∞-algebras. Using this definition one obtains a useful notion of geometric stacks in $LAff_{Sp^\Sigma}$ (though there are some complications as the étale topology is not known to be sub-canonical). The very first basic example of geometric stack is the spectra version of $\mathbb{R}Ass_V$ presented in the previous paragraph, but where V is now a finite $\mathbb{S}$-module. I will not repeat the definition here as it is very similar to the one in the linear context. We obtain this way a stack $\mathbb{R}Ass_V$, classifying associative ring structures on the spectra V, and one can prove that $\mathbb{R}Ass_V$ is a geometric stack (see [To-Ve7]). This geometric stack gives a way to approach geometrically similar questions than the one stated in [Goe-Hop].

5 Tannakian duality for Segal categories

My wish to generalize Tannakian duality to the more general setting of Segal categories comes back to the spring 1999 while I was reading *pursuing stacks*, and have been my original interest in higher category theory. My main objective was to understand in what sense affine group schemes duals to Tannakian categories are *fundamental groups*, as is usually considered in the literature. In other words, my

original question was *of which kind of 'spaces' are they the fundamental groups ?* This question was motivated by the *schematization problem* exposed in [Gr1], and the general feeling that the pro-algebraic completion of the fundamental group of topological space X is only the π_1 of a more general object, called the *schematization*, and encoding higher homotopical information. In this section I will present the general formalism of Tannakian Segal categories, for which several applications and examples will be discusses in the next two sections.

The starting point of the Tannakian theory for Segal categories is on one hand the well known analogies between Galois and Tannakian duality, and on the other hand the generalization of Galois theory to the Segal setting exposed in §3.2. I personably like to keep in mind the following informal scheme

$$\begin{array}{ccc} \{Galois\ theory:\ locally\ constant\ sheaves\} & \longrightarrow & \{Tannakian\ theory:\ local\ systems\} \\ \downarrow & & \downarrow \\ \{Segal\ Galois\ theory:\ locally\ constant\ stacks\} & \longrightarrow & \{Segal\ Tannakian\ theory:\ complexes\ of\ local\ systems\} \end{array}$$

In this last diagram, the vertical arrows represent the passage to homotopical mathematics (i.e. from categories to Segal categories), whereas the horizontal ones represent the linearization process (i.e. passing from sets to vector spaces and from homotopy types to complexes of vector spaces). The diagram also says that the coefficients of Galois theory are locally constant sheaves of sets, whereas in the Tannakian theory they are local systems of vector spaces. In the Segal setting the coefficients of Galois theory are locally constant stacks, and in the Tannakian theory are complexes of local systems (or rather complexes whose cohomology are local systems).

In this section I will describe the general notions of Segal version of the Tannakian formalism, as tensor structures, fiber functors .... I will also state the main theorem of Tannakian duality for Segal categories, which unfortunately remains a conjecture. The situation here is a bit frustrating as it seems no new ideas are really required in order to prove it, and that the standard arguments used in the proof of the usual Tannakian duality should also make sense for Segal categories. However, certain technical difficulties concerning homotopy coherences appeared during the application of Beck's theorem for Segal categories and have prevented me to find a complete proof.

The material of this section is extracted from [To3], with some slight modifications. For example, the present version of the stack of fiber functors is defined over all commutative ring spectra, whereas in [To3] I was only considering its restriction to commutative algebras, which does not seem enough in non-zero characteristic.

5.1 Tensor Segal categories

The fundamental notion is the one of symmetric monoidal Segal category, already briefly used in §2.5, and more generally the 2-Segal category of symmetric monoidal categories. It can be defined as follows.

Let $\underline{SeCat}$ be the 2-Segal category of Segal categories, as defined for example in [Hi-Si, §]. We consider Γ, the category of pointed finite sets and pointed morphisms between them, which is considered as a 2-category and therefore as a 2-Segal category. Let $\mathbb{R}\underline{Hom}(\Gamma, \underline{SeCat})$ be the 2-Segal category of (derived) morphisms of 2-Segal categories from Γ to $\underline{SeCat}$, defined using the model category structure on the category of 2-Segal precategories of [Hi-Si, §2, §11].

For any integers $1 \leq n$, we consider the finite set $[n] := \{0, \ldots, n\}$, pointed at 0, as an object of Γ. One has for any $1 \leq i \leq n$ the pointed morphisms

$$s_i : \{0, \ldots, n\} \longrightarrow \{0, 1\}$$

sending everything to 0 and i to 1. This defines, for any morphisms of 2-Segal categories $A : \Gamma \longrightarrow \underline{SeCat}$, a well defined morphism of Segal categories

$$\prod_i s_i : A([n]) \longrightarrow A([1])^n,$$

called the n-th Segal morphism (compare with [Hi-Si, §2]).

Definition 5.1 *The* 2-Segal category of symmetric monoidal Segal categories *(*$\otimes$-Segal categories *for short) is defined to be the full sub-2-Segal category of* $\mathbb{R}\underline{Hom}(\Gamma, \underline{SeCat})$ *consisting of morphisms* $A : \Gamma \longrightarrow \underline{SeCat}$ *such that*

- $A([0]) \simeq *$
- *for any integer* n *the natural morphism* $A([n]) \longrightarrow A([1])^n$ *is an equivalence of Segal categories.*

The 2-Segal category of $\otimes$*-Segal categories will be denoted by* $\otimes - \underline{SeCat}$*. For two* $\otimes$*-Segal categories* A *and* B*, the Segal category of symmetric monoidal morphisms (*$\otimes$-morphisms *for short) will be denoted by*[7]

$$\mathbb{R}\underline{Hom}^{\otimes}(A, B) := \otimes - \underline{SeCat}_{(A,B)}.$$

Finally, the underlying Segal category of a $\otimes$*-Segal category* A *is* $A([1])$*, and will be still denoted by* A*.*

[7] Here, $A_{(x,y)}$ denotes the Segal category of morphisms between two objects x and y in a 2-Segal category A.

The situation here is very much the same as the definition of Segal categories, and the monoidal structure is encoded in the diagram of Segal categories

$$\begin{array}{ccc} A([2]) & \longrightarrow & A([1]) \\ \downarrow & & \\ A([1]) \times A([1]), & & \end{array}$$

where the vertical arrow is an equivalence. This shows the existence of a morphism $-\otimes- : A\times A \longrightarrow A$, well defined in the homotopy category of Segal categories, which is the first step of the structure of a $\otimes$-Segal category A. Of course, this morphism is not enough to recover the whole A, and higher coherencies are encoded in the whole diagram over Γ. In any case, we will often use the morphism $-\otimes- : A \times A \longrightarrow A$.

Another important remark is that when A is a $\otimes$-Segal category then its homotopy category $Ho(A)$ has a natural symmetric monoidal structure. Indeed, as the functor $A \mapsto Ho(A)$ commutes with finite products, the composite functor

$$Ho(A) : \Gamma \longrightarrow \underline{SeCat} \xrightarrow{Ho(-)} \underline{Cat} \hookrightarrow \underline{SeCat}$$

still satisfies the condition to be a $\otimes$-Segal category. Of course, the structure of $\otimes$-Segal category A is more than just a symmetric monoidal structure on the homotopy category $Ho(A)$, as it encodes also higher homotopy coherences (in particular for the commutativity).

Definition 5.1 is purely in terms of Segal categories, and in practice it is very useful to have a more down to earth description of the 2-Segal category $\otimes - \underline{SeCat}$. This is possible thanks to the 2-Segal version of the strictification theorem 2.1, showing that $\otimes - \underline{SeCat}$ can be described in the following way.

- Its objects are functors
$$A : \Gamma \longrightarrow SeCat$$
from Γ to the (usual) category of Segal categories, which satisfy the following two conditions.
  - $A([0]) = *$
  - For any $n \geq 1$, the natural morphism $A([n]) \longrightarrow A([1])^n$ is an equivalence.

- Recall the existence of the model category of Segal precategories, $PrCat$ (see [Hi-Si, To-Ve1]). $\otimes$-Segal categories can therefore be considered as Γ-diagrams in $PrCat$, for which there exists a well known model category structures (equivalences and fibrations are defined levelwise, see [Hir]).

  Given two $\otimes$-Segal categories A and B, corresponding to two functors $A, B : \Gamma \longrightarrow SeCat$, the Segal category of $\otimes$-morphisms $\mathbb{R}\underline{Hom}^{\otimes}(A, B)$ is equivalent

to the Segal category of morphisms $\underline{Hom}(QA, RB)$, where QA is a cofibrant replacement of the diagram A and RB is a fibrant replacement of the diagram B.

Remark 5.2 *1.* The previous description of the 2-Segal category $\otimes$-Segal categories is actually a consequence of the existence of a model category of $\otimes$-Segal precategories, as described in [To3].

2. Our $\otimes$-Segal categories are generalization of *$ACU-\otimes$-categories* in the sense of [Sa]. We could also defined various notions of $\otimes$-Segal categories corresponding to various combination of the constraints A (associativity), C (commutativity) and U (unital). In the Segal setting the commutativity constraint is the most interesting, because it has an infinity of non-equivalent generalizations, the notion of d-fold $\otimes$-Segal categories briefly encountered in §2.5. In some sense, our present notion corresponds to the case $d = \infty$, and is the most commutativity condition one could define.

There are many interesting examples of $\otimes$-Segal categories.

- As any category is a Segal category, any $ACU - \otimes$ category ($\otimes$-category for short), in the sense of [Sa] is a $\otimes$-Segal category. Indeed, if C a $\otimes$-category one can construct a pseudo-functor $\Gamma \longrightarrow Cat$, sending $[n]$ to C^n and using the monoidal structure to define transition functors (this construction is discussed in great details in [Le2]). Strictifying this pseudo-functor gives a functor $\Gamma \longrightarrow Cat$, and therefore a functor $\Gamma \longrightarrow SeCat$ which is easily seen to satisfy the condition to be a $\otimes$-Segal category.

  Using this construction, we will always consider $\otimes$-categories as $\otimes$-Segal categories. Furthermore, this defines a natural morphism of 2-Segal categories

  $$\otimes - \underline{Cat} \longrightarrow \otimes - \underline{SeCat}$$

  which is fully faithful. The essential image of this morphism clearly consists of all $\otimes$-Segal categories whose underlying Segal category is equivalent to a category.

- Let M be a symmetric monoidal model category, in the sense of [Ho, §4.3]. Then, the Segal category LM has a natural structure of a $\otimes$-Segal category. It can be described in the following way. We consider $M^{c,1}$, the full subcategory of M of consisting of cofibrant objects and the unit (which might be non-cofibrant). Then, $M^{c,1}$ is a category with a notion of equivalences and a compatible monoidal structure. This implies that the corresponding functor $\Gamma \longrightarrow Cat$ is in fact a functor from Γ to the category of pairs (C, W) consisting of a category C and a sub-category $W \subset C$. Composing with the construction $(C, W) \mapsto L(C, W)$ one gets a functor $\Gamma \longrightarrow SeCat$ which is a

$\otimes$-Segal category. Furthermore, the underlying Segal category is $LM^{c,1}$, which is easily seen to be equivalent to LM via the natural embedding $M^{c,1} \hookrightarrow M$.

This construction gives a lot of interesting $\otimes$-Segal categories, as for example the Segal categories of complexes and of symmetric spectra.

- There is a natural morphism of 2-Segal categories $\Pi_\infty : Top \longrightarrow \underline{SeCat}$ sending a simplicial set to its Segal fundamental groupoid (see [Hi-Si, §2], where our Π_∞ is denoted by $\Pi_{1,se}$), and which identifies Top with the full 2-Segal subcategory of $\underline{SeCat}$ consisting of Segal groupoids. Using this identification, one sees that there is an equivalence between the 2-Segal category of Γ-spaces and the 2-Segal category of $\otimes$-Segal categories whose underlying Segal category is a Segal groupoid. Therefore, this provides an equivalence between the 2-Segal category of connective spectra and the 2-Segal category of $\otimes$-Segal groupoids.

We now come to the cental definition of a tensor Segal category, an analog of the notion of tensor category in the Segal setting.

Definition 5.3 *A* tensor Segal category *is a $\otimes$-Segal category A satisfying the following two conditions.*

- *The underlying Segal category A is stable (in the sense of Def. 2.5).*
- *For any object $x \in A$, the morphism of Segal categories $x \otimes - : A \longrightarrow A$ is exact.*

For any tensor Segal categories A and B, we denote by $\mathbb{R}\underline{Hom}^{\otimes}_{ex}(A,B)$ the full sub-Segal category of $\mathbb{R}\underline{Hom}^{\otimes}(A,B)$ consisting of exact morphisms. Objects in $\mathbb{R}\underline{Hom}^{\otimes}_{ex}(A,B)$ will be called tensor morphisms from A to B.

Tensor Segal categories and $\mathbb{R}\underline{Hom}^{\otimes}_{ex}$ form a 2-Segal category denoted by $\underline{TenSeCat}$.

Tensor Segal categories essentially comes from the following example. Let M be a symmetric monoidal model category. We also assume that the model category M is stable (in the sense of [Ho, §7]). Then, the $\otimes$-Segal category LM is a tensor Segal category. Indeed, clearly LM is stable Segal category (see Thm. 2.7 (4)). Furthermore the existence of the adjunction equivalence $Map_M(x \otimes^{\mathbb{L}} y, z) \simeq Map_M(x, \mathbb{R}\underline{Hom}(y,z))$, implies that $x \otimes^{\mathbb{L}} -$ commutes with homotopy colimits in M. This is equivalent to condition (2) of Def. 5.3.

Remark 5.4 As a tensor Segal category A is on one hand a stable Segal category, and the other hand a $\otimes$-Segal category, the homotopy category $Ho(A)$ is endowed with a natural triangulated structure and a symmetric monoidal structure. Of course, these two structures are compatible, making $Ho(A)$ into a tensor triangulated category.

Let us now consider the category of symmetric spectra Sp^Σ, endowed with its positive model category structure described in [Shi, Prop. 3.1]. The model category Sp^Σ has a natural monoidal structure, $-\wedge-$, making into a symmetric monoidal model category. Let A be a commutative (unital and associative) monoid in Sp^Σ (we will simply say that A is a commutative ring spectrum), and $A-Mod$ be its category of modules. From [Shi, Thm. 3.2] we know that $A-Mod$ is a symmetric monoidal model category. Furthermore, this model category is clearly stable. Therefore, the Segal category $L(A-Mod)$ is then a tensor Segal category. The tensor Segal categories $L(A-Mod)$ are very important, as they are the Segal version of the tensor categories of modules over some commutative rings, and actually they behave even better as their monoidal structure is always exact. Furthermore, it is important to notice that if k is a commutative ring, and Hk is the corresponding Eilenberg-MacLane commutative ring spectrum, the $L(Hk-Mod)$ is equivalent (as a tensor Segal category) to $LC(k)$, where $C(k)$ is the symmetric monoidal model category of complexes of k-modules.

Let $A\longrightarrow B$ be a morphism of commutative ring spectra, one has a base change morphism

$$-\wedge^{\mathbb{L}}_A B : L(A-Mod)\longrightarrow L(B-Mod),$$

which is a morphism of tensor Segal categories. If we denote by $Comm(Sp^\Sigma)$ the model category of commutative ring spectra (see [Shi, Thm. 3.2]), this construction defines a morphism of 2-Segal categories

$$LComm(Sp^\Sigma)\longrightarrow \underline{TensSeCat}.$$

The first fundamental conjecture is the following.

Conjecture 5.5 *The morphism of* 2*-Segal categories*

$$\begin{array}{ccc} L(Comm(Sp^\Sigma)) & \longrightarrow & \underline{TensSeCat} \\ A & \mapsto & L(A-Mod) \\ (A\to B) & \mapsto & -\wedge^{\mathbb{L}}_A B \end{array}$$

is fully faithful.

Of course, the natural approach to prove the conjecture 5.5 is by defining a morphism in the other way, sending a tensor Segal category A to $A_{(1,1)}$, where 1 is the unit of the tensor structure. However, to endow $A_{(1,1)}$ with a natural structure of commutative ring spectrum requires to solve some homotopy coherence problems which do not seem so obvious. In any case, the conjecture seems to me clearly correct.

I would also like to mention that conjecture 5.5 is surely false if one considers the tensor triangulated categories $Ho(A-Mod)$ instead of the Segal categories $L(A-Mod)$, as it might exists two non-equivalent commutative ring spectra

with equivalent tensor triangulated categories of modules (similar examples for non-commutative ring spectra are given in [Schw-Shi, Example. 3.2.1]).

Another important conjecture is a comparison between the theories of commutative ring spectra and of commutative monoids in the tensor Segal category LSp^{Σ}. Let FS be the category of finite sets and all morphisms between them. The disjoint union makes FS into a symmetric monoidal category and therefore into a $\otimes$-Segal category. For a $\otimes$-Segal category A, the Segal category of commutative monoids in A is defined to be

$$Comm(A) := \mathbb{R}\underline{Hom}^{\otimes}(FS, A).$$

In [To3] I have constructed a natural morphisms of Segal categories

$$L(Comm(Sp^{\Sigma})) \longrightarrow Comm(LSp^{\Sigma}),$$

where $Comm(Sp^{\Sigma})$ is as before the model category of commutative ring spectra.

Conjecture 5.6 *The natural morphism*

$$L(Comm(Sp^{\Sigma})) \longrightarrow Comm(LSp^{\Sigma})$$

is an equivalence of Segal categories.

Conjecture 5.6 is the key result in order to have a Segal category interpretation of the theory of ring spectra. It has also a sense when Sp^{Σ} is replaced by any symmetric monoidal model category and $Comm(Sp^{\Sigma})$ by the model category of E_{∞}-algebra in M, and can be seen as a monoidal analog of the strictification theorem Thm. 2.1. I also explains why the notion of commutative ring spectra is the *right one*, as it corresponds to *commutative monoids in the monoidal ∞-category of spectra*.

We finish by the Segal version of the notion of rigidity (see e.g. [Br, Sa]).

Definition 5.7 *Let A be a $\otimes$-Segal category A.*

- *We say that A is* closed *if for any two objects x and y in A, the morphism of Segal categories*

$$\begin{array}{ccc} A^{op} & \longrightarrow & Top \\ z & \mapsto & A_{(z\otimes x, y)} \end{array}$$

  is representable by an object $\mathbb{R}\underline{Hom}(x, y) \in A$ (in the sense explained in §2.1).

- *We say that A is* rigid *if it is closed and furthermore if for any object x in A, the natural morphism in $Ho(A)$*

$$\mathbb{R}\underline{Hom}(x, 1) \otimes x \longrightarrow \mathbb{R}\underline{Hom}(x, x)$$

  is an isomorphism in $Ho(A)$.

An important remark is that when M is a symmetric monoidal model category then the $\otimes$-Segal category LM is always closed. Indeed, the existence of derived Hom objects in M (see [Ho, §4.3]) implies the existence of the objects $\mathbb{R}\underline{Hom}(x,y) \in LM$.

In a rigid $\otimes$-Segal category A one always has the famous formula

$$\mathbb{R}\underline{Hom}(x \otimes x', y \otimes y') \simeq \mathbb{R}\underline{Hom}(x,y) \otimes \mathbb{R}\underline{Hom}(x',y'),$$

and in particular

$$\mathbb{R}\underline{Hom}(x,y) \simeq \mathbb{R}\underline{Hom}(x,1) \otimes y.$$

5.2 Stacks of fiber functors

The right setting to state the Tannakian duality for Segal categories is BNAG, exposed in §4.4. We refer to this paragraph for the general notions, and we start by defining the Segal version of the well known adjunction between tensor categories and stacks using the central notion fiber functors.

We consider the model category $Comm(Sp^{\Sigma})$ of commutative, associative and unital monoids in the category of symmetric spectra Sp^{Σ}. By [Shi, Thm. 3.2], we know that $Comm(Sp^{\Sigma})$ is endowed with a model category structure for which fibrations and equivalences are defined on the level of underlying spectra. Objects in $Comm(Sp^{\Sigma})$ will simply be called *commutative ring spectra*. We consider the Segal category $LComm(Sp^{\Sigma})$ as well as its opposite category that we will denote by

$$LComm(Sp^{\Sigma})^{op} := LAff_{Sp^{\Sigma}}.$$

We endow the Segal category $LAff_{Sp^{\Sigma}}$ with the following topology. A finite family of morphisms of commutative ring spectra $\{A \longrightarrow B_i\}_{i\in I}$ is called a *strongly ffqc covering* if it satisfies the following conditions.

- The family of morphisms of affine schemes $\{Spec\, H^0(B_i) \longrightarrow Spec\, H^0(A)\}_{i\in I}$ is faithfully flat.
- For any i, the natural morphism of $H^0(B_i)$-modules $H^*(A) \otimes_{H^0(A)} H^0(B_i) \longrightarrow H^*(B_i)$ is an isomorphism.

One checks that this defines a Segal topology on the Segal category $LAff_{Sp^{\Sigma}}$ in the sense of Def. 3.2, and therefore on has an associated Segal category of stacks $LAff^{\sim,sffqc}_{Sp^{\Sigma}}$ (*sffqc* stands for *strongly faithfully flat and quasi-compact*). The Segal topology $sffqc$ is sub-canonical and therefore the Yoneda embedding gives a fully faithful morphism of Segal categories

$$h : LAff_{Sp^{\Sigma}} \longrightarrow LAff^{\sim,sffqc}_{Sp^{\Sigma}}.$$

The image of a commutative ring spectra A by h is as usual denoted by $\mathbb{R}Spec\, A$.

The Segal site $LAff_{Sp^\Sigma}$ has a natural stack in tensor Segal categories $LParf$, which is a BNAG version of the usual stack of vector bundles. It is defined in the following way.

For any commutative ring spectrum A, one has a closed $\otimes$-Segal category $L(A-Mod)$ of A-modules. One can consider the full sub-Segal category $L(A-Mod)^{rig}$ consisting of *rigid objects*. Recall here that an object x in a closed $\otimes$-Segal category T (in the sense of definition Def. 5.7 (1)) is *rigid* if the natural morphism $\underline{Hom}(x,1)\otimes x \longrightarrow \underline{Hom}(x,x)$ is an isomorphism in $Ho(T)$. Rigid objects in $L(A-Mod)$ are precisely the A-modules which are *strongly dualizable* in the sense of [EKMM], and therefore consist of all retract of finite cell A-modules. One also sees that rigid objects in $L(A-Mod)$ are precisely the finitely presented objects (in the sense of Segal category). The Segal category $L(A-Mod)^{rid}$ is called the Segal category of perfect A-modules, and is denoted by $LParf(A)$. Clearly, $LParf(A)$ is a rigid tensor Segal category.

For a morphism of commutative ring spectra $A \longrightarrow B$, one has a base change morphism

$$\begin{array}{ccc} LParf(A) & \longrightarrow & LParf(B) \\ M & \mapsto & M \wedge^{\mathbb{L}}_A B, \end{array}$$

defining (after some standard strictification procedure) a morphism of 2-Segal categories

$$\begin{array}{ccc} LAff^{op}_{Sp^\Sigma} & \longrightarrow & \underline{TenSeCat}^{rig} \\ A & \mapsto & LParf(A) \\ (A\to B) & \mapsto & -\wedge^{\mathbb{L}}_A B, \end{array}$$

from the Segal site $LAff^{op}_{Sp^\Sigma}$ to the 2-Segal category of rigid tensor Segal categories. This morphism is in fact a *stack in tensor Segal categories*, in the sense that the underlying pre-stack of Segal categories is a stack (over the Segal site $LAff^{op}_{Sp^\Sigma}$).

Definition 5.8 *The* stack of perfect modules *is the stack of tensor Segal categories defined above. It is denoted by $LParf$.*

The stack $LParf$ can be used in order to define a morphism from the 2-Segal category of stacks $LAff^{\sim,sffqc}_{Sp^\Sigma}$ to the 2-Segal category $\underline{TenSeCat}^{rig}$ of rigid tensor Segal categories. For a stack $F \in LAff^{\sim,sffqc}_{Sp^\Sigma}$, one can consider the Segal category $LParf(F) := \mathbb{R}\underline{Hom}(F, LParf)$. This Segal category is naturally a rigid tensor Segal category, and therefore one has a morphism of 2-Segal categories

$$\begin{array}{ccc} (LAff^{\sim,sffqc}_{Sp^\Sigma})^{op} & \longrightarrow & \underline{TenSeCat}^{rig} \\ F & \mapsto & LParf(F). \end{array}$$

This morphism possesses a left adjoint

$$\begin{array}{ccc} \underline{TenSeCat}^{rig} & \longrightarrow & (LAff^{\sim,sffqc}_{Sp^\Sigma})^{op} \\ T & \mapsto & FIB(T), \end{array}$$

where $FIB(T) \in LAff^{\sim,sffqc}_{Sp^\Sigma}$ is the stack of fiber functors on T defined as follows. For any commutative ring spectra A one defines

$$FIB(T)(A) := |\mathbb{R}\underline{Hom}^{\otimes}_{ex}(T, LParf(A))|,$$

the geometric realization of the Segal category of tensor morphisms from T to $LParf(A)$[8]. One has the following adjunction equivalence for a stack $F \in LAff^{\sim,sffqc}_{Sp^\Sigma}$ and a rigid tensor Segal category T

$$\mathbb{R}\underline{Hom}(F, FIB(T)) \simeq |\mathbb{R}\underline{Hom}^{\otimes}_{ex}(T, LParf(F))|,$$

saying that $T \mapsto FIB(T)$ is the left adjoint to $F \mapsto LParf(F)$.

The situation can also be generalized over a base commutative ring spectra A in the following way. Instead of the 2-Segal category $\underline{TenSeCat}^{rig}$, of rigid tensor Segal categories, one consider $A-\underline{TenSeCat}^{rig}$, the 2-Segal category $LParf(A)/\underline{TenSeCat}^{rig}$, of rigid tensor Segal categories under $LParf(A)$. The 2-Segal category $A-\underline{TenSeCat}^{rig}$ will be called the 2-*Segal category of rigid A-tensor Segal categories*. Its objects are simply rigid tensor Segal categories T together with a tensor morphism $LParf(A) \longrightarrow T$. For two objects $LParf(A) \longrightarrow T$ and $LParf(A) \longrightarrow T'$ the Segal category of morphisms from T to T' in $LParf(A)/\underline{TenSeCat}^{rig}$ sits naturally in a homotopy cartesian square

$$\begin{array}{ccc} \mathbb{R}\underline{Hom}^{\otimes}_{A-ex}(T,T') := A - \underline{TenSeCat}^{rig}_{(T,T')} & \longrightarrow & \mathbb{R}\underline{Hom}^{\otimes}_{ex}(T,T') \\ \downarrow & & \downarrow \\ \bullet & \longrightarrow & \mathbb{R}\underline{Hom}^{\otimes}_{ex}(LParf(A),T'). \end{array}$$

In other words $\mathbb{R}\underline{Hom}^{\otimes}_{A-ex}(T,T')$ is the (homotopy) fiber of

$$\mathbb{R}\underline{Hom}^{\otimes}_{ex}(T,T') \longrightarrow \mathbb{R}\underline{Hom}^{\otimes}_{ex}(LParf(A),T')$$

at the point corresponding to the structural morphism $LParf(A) \longrightarrow T'$.

For a rigid A-tensor Segal category $LParf(A) \longrightarrow T$, one can define its stack of fiber functors $FIB_A(T)$, which naturally lives in the Segal category $LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A$

[8]Note that the rigidity condition on T implies that $\mathbb{R}\underline{Hom}^{\otimes}_{ex}(T, LParf(A))$ is actually a Segal groupoid, which justifies considering its geometric realization.

of stacks over $\mathbb{R}Spec\, A$[9]. This defines a morphism of 2-Segal categories

$$\begin{array}{ccc} A-\underline{TenSeCat}^{rig} & \longrightarrow & (LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A)^{op} \\ T & \mapsto & FIB_A(T), \end{array}$$

which also has a right adjoint

$$\begin{array}{ccc} (LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A)^{op} & \longrightarrow & A-\underline{TenSeCat}^{rig} \\ F & \mapsto & LParf_A(F) := (LParf(A) \to LParf(F))\,. \end{array}$$

Definition 5.9 *For a rigid A-tensor Segal category T, the* stack of fiber functors (over A) *is the stack $FIB_A(T) \in LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A$ defined above.*

The adjunction $(T \mapsto FIB_A(T), F \mapsto LParf_A(F))$ is the fundamental object of study, and the Tannakian duality for Segal categories describes conditions on stacks and A-tensor Segal categories for which this adjunction induces a one-to-one correspondence.

5.3 The Tannakian duality

We let A be a base commutative ring spectra. For example A could be of the form Hk, for a commutative ring k, which will be our main example of applications.

In this part I will introduce the notions of *A-Tannakian Segal categories* and of *Tannakian (Segal) gerbes over A*, as well as a conjecture stating that the constructions $F \mapsto LParf_A(F)$ and $T \mapsto FIB_A(T)$ induces an equivalence between them. This conjecture is the Segal version of the Tannakian duality theorem giving an equivalence between affine gerbes and Tannakian categories as stated in [De1, Sa]. However, some technical difficulties have prevented me to actually prove this conjecture, thought I am convinced it is correct.

I start with the definition of affine gerbes in the Segal setting. For this I will need the following notion of morphisms of positive Tor dimension between commutative

[9] The Segal category $LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A$ is equivalent to the category of stacks over the Segal site of commutative A-algebras. In other words, with the notations of §4 one has

$$LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A \simeq LAff^{\sim,sffqc}_{A-Mod}$$

where $A-Mod$ is the model category of A-modules, and $sffqc$ is the induced strongly flat topology defined as for commutative ring spectra. In particular, objects in $LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A$ can be considered as functors

$$F : A/Comm(Sp^\Sigma) \longrightarrow SSet$$

from the category of commutative A-algebras to the category of simplicial sets.

ring spectra. For a morphism $f : B \longrightarrow B'$ of commutative ring spectra, one says that f is of *positive Tor dimension*[10] if for any B-module M,

$$(\pi_i(M) = 0 \; \forall \; i > 0) \Rightarrow \left(\pi_i(M \wedge^{\mathbb{L}}_B B') = 0 \; \forall \; i > 0\right).$$

Furthermore, one says that $f : B \longrightarrow B'$ is *faithfully of positive Tor dimension* if it is of positive Tor dimension and if moreover the base change functor

$$- \wedge^{\mathbb{L}}_B B' : Ho(B - Mod) \longrightarrow Ho(B' - Mod)$$

is conservative (i.e. $M \simeq *$ if and only if $M \wedge^{\mathbb{L}}_B B' \simeq *$).

If $f : B \longrightarrow B'$ is a strongly flat morphism in the sense explained in the last paragraph, then one has $\pi_i(M \wedge^{\mathbb{L}}_B B') \simeq \pi_i(M) \otimes_{\pi_0(B)} \pi_0(B')$, and therefore f is of positive Tor dimension. Moreover, if $k \longrightarrow k'$ is a morphism of commutative rings then $Hk \longrightarrow Hk'$ is of positive Tor dimension if and only if k' is flat over k. However, the notion of positive Tor dimension morphisms is much weaker than of strongly flat morphisms, as there exists positive Tor dimension morphisms $Hk \longrightarrow B'$, where k is commutative ring but B' has non-trivial negative homotopy groups.

As usual, the notion of positive Tor dimension morphisms extends to the case of representable morphisms of stacks.

Definition 5.10 *A stack $F \in LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A$, is an* affine (Segal) gerbe *if it satisfies the following two conditions.*

- *The stack F is locally non-empty (i.e. there exists a $sffqc$ covering $A \longrightarrow A'$ such that $F(A') \neq \emptyset$).*
- *For any commutative A-algebra B, and any two morphisms of stacks $x, y : \mathbb{R}Spec\, B \longrightarrow F$, the stack of path from x to y*

  $$\Omega_{x,y}F := \mathbb{R}Spec\, B \times_{x,F,y} \mathbb{R}Spec\, B$$

  is representable and faithfully of positive Tor dimension over $\mathbb{R}Spec\, B$ (i.e. there exists a morphism which is faithfully of positive Tor dimension $B \longrightarrow C$ such that $\Omega_{x,y}F \simeq \mathbb{R}Spec\, C$).

An affine (Segal) gerbe F is said to be neutral *if there exists a morphism of stacks $* \longrightarrow F$.*

Clearly, a general affine gerbe over A is locally for the positive Tor dimension topology of the form BG for G an affine group stack of positive Tor dimension over A. This last description is much closer to the usual notion of affine gerbe encountered in algebraic geometry, which are stacks locally equivalent to classifying stacks

[10] There are no mistakes here, has for a complex of abelian groups E, viewed as a symmetric spectra HE, one has $\pi_i(HE) \simeq H^{-i}(E)$.

of flat affine group schemes (see the next paragraph for more about the comparison with the usual notion).

We are now ready to define *Tannakian (Segal) gerbes*, which are affine gerbes satisfying some cohomological conditions, as well as *Tannakian Segal categories*.

Definition 5.11 • *A morphism $f : F \longrightarrow F'$ in $LAff_{Sp^\Sigma}^{\sim,sffqc}$ is a* P-equivalence *if the induced morphism of tensor Segal categories*

$$f^* : LPar f(F') \longrightarrow LPar f(F)$$

is an equivalence.

- *A stack $F \in LAff_{Sp^\Sigma}^{\sim,sffqc}$, is* P-local*, if for any P-equivalence of stacks $G \longrightarrow G'$, the induced morphism*

$$(LAff_{Sp^\Sigma}^{\sim,sffqc})_{(G',F)} \longrightarrow (LAff_{Sp^\Sigma}^{\sim,sffqc})_{(G,F)}$$

is an equivalence of simplicial sets.

- *A stack $F \in LAff_{Sp^\Sigma}^{\sim,sffqc}/\mathbb{R}Spec\, A$, is a* A-Tannakian (Segal) gerbe *if it is an affine gerbe over A, and if the stack F is P-local.*

- *A A-tensor Segal category T is a* A-Tannakian Segal category *if it is equivalent to some $LPar f_A(F)$ for F a A-Tannakian gerbe. A A-tensor Segal category is* neutral *if furthermore the A-Tannakian gerbe F above can be chosen to be neutral.*

Remark 5.12 When F is a A-Tannakian Segal gerbe, the tensor Segal category $LPar f_A(F)$ comes equipped with some kind of t-structure. Indeed, one can chose a $sffqc$ covering $A \to A'$ and a point $x : \mathbb{R}Spec\, A' \longrightarrow F$, and define an object $E \in LPar f_A(F)$ to be *positive* if $x^*(E)$ is a positive A'-module (i.e. $\pi_i(M) = 0$ for all $i > 0$). The fact that this definition is independent of the choice of x uses that any two points of F are locally equivalent up to a positive Tor dimension covering.

The main conjecture is the following duality statement.

Conjecture 5.13 *1. For any A-Tannakian Segal category T, the natural morphism*

$$T \longrightarrow LPar f_A(FIB_A(T))$$

is an equivalence of A-tensor Segal categories.

2. A rigid A-tensor Segal category T is Tannakian if it satisfies the following two conditions.

(a) The structural morphism $LParf(A) \longrightarrow T$ *induces an equivalence*

$$LParf(A)_{(1,1)} \longrightarrow T_{(1,1)},$$

where 1 *denotes the unit of the* $\otimes$*-structures.*

(b) There exists a $sffqc$ *covering* $A \to A'$*, and a* A*-tensor morphism*

$$\omega : T \longrightarrow LParf(A')$$

satisfying the following two conditions.

i. The extension of ω *to the Segal categories of* Ind*-objects*

$$\overline{\omega} : Ind(T) \longrightarrow Ind(LParf(A')) \simeq L(A' - Mod)$$

is conservative (i.e. $\overline{\omega}(M) \simeq *$ *if and only if* $M \simeq *$*).*

ii. Let $Ind(T)_{\geq 0}$ *(resp.* $Ind(T)_{<0}$*) be the full sub-Segal category of* $Ind(T)$ *consisting of objects* E *such that* $\pi_i(\overline{\omega}(E)) = 0$ *for all* $i \geq 0$ *(resp. for all* $i < 0$*). Then, and object* $E \in Ind(T)$ *lies in* $Ind(T)_{\geq 0}$ *if and only if for any* $F \in Ind(T)_{<0}$ *one has* $T_{(F,E)} \simeq *$*.*

Clearly, conjecture 5.13 implies that the two morphisms of 2-Segal categories FIB_A and $LParf_A$ induce an equivalence between the 2-Segal category of A-Tannakian Segal categories and the 2-Segal category of A-Tannakian gerbes.

Remark 5.14

- Condition $(2-b-ii)$ of conjecture 5.13 defines some kind of t-structure on the stable Segal category $Ind(T)$, and it could very well be that this additional structure is an important part of data for Tannakian Segal categories that I have been neglecting up to now. For example it could very well be that the equivalence $T \longrightarrow LParf_A(FIB_A(T))$ of 5.13 (1) is only correct if one take into account the natural t-structure on $Ind(T)$, and if one replaces $FIB_A(T)$ by its sub-stack of *t-positive* fiber functors (i.e. the one that preserve positive objects). I will however stay with the conjecture 5.13 as is, as it seems that for $A = Hk$ and k a field any fiber functor is automatically t-positive.

- The conjecture 5.13 is quite general, and I am not sure it is so important to have it for a general base commutative ring spectra A. Up to now, I only consider it serious when $A = Hk$ for some commutative ring k.

To finish this part I a going to describe the general steps for a proof of the main point of conjecture 5.13 in the neutral case.

Let me start by some linear algebra notions in the context of commutative ring spectra.

- A H_∞-Hopf A-algebra is a co-simplicial diagram of commutative A-algebras

$$B_* : \Delta \longrightarrow A/Comm(Sp^\Sigma)$$

such that $B_0 = A$, and for each n the natural morphism (the dual version of the n-th Segal morphism)

$$\underbrace{B_1 \wedge^{\mathbb{L}}_A B_1 \cdots \wedge^{\mathbb{L}}_A B_1}_{n\ times} \longrightarrow B_n$$

is an equivalence. Clearly, H_∞-Hopf A-algebras correspond via the $\mathbb{R}Spec$ functor to affine group stacks in $LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, A$.

- Any H_∞-Hopf A-algebra B_* has a rigid A-tensor Segal category of perfect (or rigid) comodules $L(B_* - Comod^{rig})$, defined as the limit (of Segal categories)

$$L(B_* - Comod^{rig}) \longrightarrow Holim_{n\in Delta} LPar f(B_n).$$

Clearly, one has a natural equivalence

$$LPar f(B\mathbb{R}Spec\, B_*) \simeq L(B_* - Comod^{rig}),$$

where $B\mathbb{R}Spec\, B_* := Hocolim_{n\in\Delta^{op}}\mathbb{R}Spec\, B_n$ is the stack whose loop stack is the affine group scheme $\mathbb{R}Spec\, B_1$. In the same way, one has a full (non-rigid) A-tensor Segal category of comodules

$$L(B_* - Comod) \longrightarrow Holim_{n\in Delta} L(B_n - Mod).$$

Now, let T be a rigid A-tensor Segal category satisfying the conditions of 5.13 (b), but with $A' = A$ (this is the neutral case). We first consider the induced morphism on the Segal categories of Ind-objects

$$\overline{\omega} : Ind(T) \longrightarrow L(A - Mod),$$

which is a morphism of non-rigid A-tensor Segal categories. By some general principle, the morphism $\overline{\omega}$ have a right adjoint p (this follow from the fact that it commutes with colimits and that $Ind(T)$ has a small sets of small generators). We next consider the A-module $B := \overline{\omega}(p(A))$, which is expected to have a natural structure of an H_∞-Hopf A-algebra. The comultiplication is given by the adjunction morphism

$$B = \overline{\omega}p(A) \longrightarrow B \wedge^{\mathbb{L}}_A B \simeq \overline{\omega}p\overline{\omega}p(A)$$

induced by $Id \to p\overline{\omega}$, and the multiplication is given by using the natural morphism $p(A)\otimes p(A) \to p(A)$ adjoint to $\overline{\omega}p(A)\otimes\overline{\omega}p(A) \to A\wedge^{\mathbb{L}}_A A \simeq A$ (Though this structure seems clear from an heuristic point of view, controlling all the homotopy coherences

is a problem). Furthermore, any object $M \in Ind(T)$ gives rise to a B_*-comodule $\overline{\omega}(M)$, via the co-action

$$\overline{\omega}(M) \longrightarrow B \wedge^{\mathbb{L}}_A \overline{\omega}(M) \simeq \overline{\omega}p\overline{\omega}(M),$$

induced by the adjunction $Id \to p\overline{\omega}$. One should show this way that the adjunction $(\overline{\omega}, p)$ induces a new adjunction

$$\overline{\omega} : Ind(T) \longrightarrow L(B_* - Comod) \qquad Ind(T) \longleftarrow L(B_* - Comod) : p.$$

A Segal version of the Barr-Beck theorem should now be applied, thanks to the conservative property of $\overline{\omega}$, which shows that this last adjunction is an equivalence. This equivalence identifies the sub-Segal categories of rigid objects.

In conclusion, we have shown that for any rigid A-tensor Segal category T which satisfies the conditions (b) of conjecture 5.13 there exists an equivalence between $Ind(T)$ and $L(B_* - Comod)$ which transforms the fiber functor $\overline{\omega}$ into the forgetful functor $L(B_* - Comod) \longrightarrow L(A - Mod)$. We can therefore replace T by $L(B_* - Comod^{rig})$, which implies that T is of the form $LPar f(F)$ for F the classifying stack of the group stack $\mathbb{R}Spec\, B$. Finally, the second assertion on $\overline{\omega}$ will imply that the A-algebra $B = \overline{\omega}p(A)$ is faithfully of positive Tor dimension, showing that F is an affine gerbe. This would be the main point for a proof of conjecture 5.13, and the remaining statements should follow quite formally from it.

5.4 Comparison with usual Tannakian duality

We now suppose that k is a commutative ring, and that $A = Hk$ is its Eilenberg-MacLane commutative ring spectra[11]. On one hand one has the Segal topos $LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, Hk$, whose objects can be seen as functors

$$F : Hk/Comm(Sp^\Sigma) \longrightarrow SSet$$

from the category of commutative Hk-algebras to the category of simplicial sets (and satisfying the usual stack conditions). One the other hand one has the usual Grothendieck site $(Aff/k, ffqc)$, of affine k-schemes with the faithfully flat topology, and its associated Segal topos $LAff^{\sim,ffqc}_k$. Sending a commutative k-algebra k' as a commutative Hk-algebra Hk', induces a well defined morphism of Segal categories (it is not a geometric morphism of Segal topoi)

$$i : LAff^{\sim,ffqc}_k \longrightarrow LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, Hk,$$

which on affine objects sends $Spec\, k'$ to $\mathbb{R}Spec\, Hk'$. This morphism is actually fully faithful, and possesses a right adjoint

$$h^0 : LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, Hk \longrightarrow LAff^{\sim,ffqc}_k$$

[11] In this situation A-tensor Segal categories will simply be called k-tensor Segal categories.

defined for a stack $F \in LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, Hk$ by the restriction, $h^0(F)(k') := F(Hk')$.

The very first observation is that if F is an affine gerbe in the sense of [De1], then $i(F) \in LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, Hk$ is an affine Segal gerbe in the sense of Def. 5.10, which is neutral if and only if F is. This way, affine gerbe in the sense of [De1] form a full sub-Segal category of the Segal category of affine Segal gerbes. Our notion of affine Segal gerbe is therefore a reasonable generalization of the usual notion.

Let us now suppose that k is a field, and let T be a Tannakian category over k in the sense of [De1]. We let $\mathcal{G}$ be the gerbe of fiber functors on T, which is an object in $LAff^{\sim,ffqc}_k$.

We consider $C^b(T)$, the category of bounded complexes in T, and we let $LParf(T) := LC^b(T)$ be the Segal category obtained by localizing $C^b(T)$ along the quasi-isomorphisms (and which we will call the Segal category of perfect complexes in T). As T is Tannakian over k, there is a natural k-tensor functor $k - Vect \longrightarrow T$, sending a k-vector space V to the external product $V \otimes 1$, where $1 \in T$ is the unit (as usual, for $E, F \in T$, $V \otimes E$ is determined by the adjunction formula $Hom_T(V \otimes E, F) \simeq Hom_{k-Vect}(V, Hom_T(E, F))$). Passing to the Segal categories of perfect complexes one finds a morphism of $\otimes$-Segal categories

$$LParf(k - Vect) \simeq LParf(Hk) \longrightarrow LParf(T),$$

making $LParf(T)$ into a rigid k-tensor Segal category.

Resuming the notations, one has a k-Tannakian category T together with its gerbe of fiber functors $\mathcal{G}$, and a rigid k-tensor Segal category $LParf(T)$ together with its stack of fiber functors $FIB_k(LParf(T))$. The claim is then the following.

Proposition 5.15 *The k-tensor Segal category $LParf(T)$ is Tannakian, and furthermore one has a natural equivalence of stacks in $LAff^{\sim,sffqc}_{Sp^\Sigma}/\mathbb{R}Spec\, Hk$*

$$i(\mathcal{G}) \simeq FIB_k(LParf(T)).$$

The main corollary of proposition Prop. 5.15 is the following, showing that our notion of k-Tannakian Segal category is a reasonable generalization of the usual notion.

Corollary 5.16 *The construction $T \mapsto LParf(T)$ defines a fully faithful morphism from the 2-Segal category of k-Tannakian categories to the 2-Segal category of k-Tannakian Segal categories.*

One can also go further and characterize the image of the morphism $T \mapsto LParf(T)$. Indeed, if T' is any k-Tannakian Segal category, then T' has a natural t-structure defined as in 5.13 $(b-i)$ whose heart will be denoted by $\mathcal{H}(T')$. The

heart $\mathcal{H}(T')$ is always a k-Tannakian category, and there exists a natural morphism of k-tensor Segal categories

$$LPar f(\mathcal{H}(T')) \longrightarrow T'.$$

Then, T' is of the form $LPar f(T)$ for some k-Tannakian category T if and only if this last morphism is an equivalence (and then furthermore one has $T \simeq \mathcal{H}(T')$).

In the general situation, but still when k is a field, for any k-Tannakian Segal category T, one has its heart $\mathcal{H}(T)$, which is a k-Tannakian category, together with the natural morphism of k-tensor Segal categories

$$j : LPar f(\mathcal{H}(T)) \longrightarrow T.$$

It is not hard to show that after passing to Ind-objects, j has a right adjoint

$$q : Ind(T) \longrightarrow Ind(LPar f(\mathcal{H}(T))) \simeq LC(Ind(\mathcal{H}(T))),$$

where $LC(Ind(\mathcal{H}(T)))$ is the Segal category of complexes in the abelian category of Ind-objects $Ind(\mathcal{H}(T))$. Using the adjunction (j, q) one should then construct a commutative monoid $A := q(1)$ in $LC(Ind(\mathcal{H}(T)))$, and an equivalence between $Ind(T)$ and the Segal category $A-Mod$ of A-modules in $LC(Ind(\mathcal{H}(T)))$. Condition $(2-ii)$ of conjecture 5.13 would then implies that A is cohomologically concentrated in non-negative degrees. This construction would give a structure theorem for k-Tannakian Segal categories when k is a field, that we state as a conjecture.

Conjecture 5.17 *Let k be a field and T be a k-Tannakian Segal category. Then there exists a k-Tannakian category $\mathcal{H}(T)$, and a non-negatively graded E_∞-algebra A in the k-tensor category $Ind(\mathcal{H}(T))$, such that T is equivalent to the Segal category of E_∞-modules over A in $Ind(\mathcal{H}(T))$.*

One of the main consequence of conjecture 5.17 would be that any Tannakian Segal category over a field has a nice model as a model category, and that they can all be constructed as Segal categories of modules over an E_∞-algebra in a usual Tannakian category. Essentially, this means that Tannakian Segal categories over fields are essentially the same thing as E_∞-algebras equipped with an action of an affine group scheme G. Geometrically, the affine group scheme G is the fundamental group of the associated Tannakian Segal gerbe, whereas the E_∞-algebra A is a model for the homotopy type of its universal covering (i.e. is its E_∞-algebra of cohomology). This last picture actually appears in practice, where one constructs directly certain equivariant E_∞-algebras without using the notion of Tannakian Segal categories (see for example [Ha, Ka-Pa-To1, Ol]).

6 Homotopy types of algebraic varieties

The cohomology groups of algebraic varieties are endowed with additional structures reflecting the algebraic nature of the space (e.g. Hodge structures, Galois actions, crystal structures ...). More generally, one expects that algebraic varieties have not only cohomology groups but full homotopy types also having additional structures extending the one on cohomology. In this section I will present some of my works around the very basic question: *How to define interesting homotopy types of algebraic varieties, and what are their additional structures ?* The references for this section are [Ka-Pa-To1, To2, To3].

I have been considering seriously only the case of complex algebraic varieties. Indeed, in the complex situation, algebraic varieties has an underlying topological space of complex point (with the analytic topology) and therefore a given homotopy type. Of course, as Hodge structures does not exist directly on the integral cohomology groups but only after tensoring with $\mathbb{C}$, one could not expect the topological homotopy type of a complex variety to have a reasonable Hodge structure. One first problem was therefore to extract from the topological homotopy type the part that has to do with algebraic geometry and that one can endow with an additional Hodge structure. One possible answer to this is the solution to the schematization problem of Grothendieck presented below. I will then describe how the schematization of a complex smooth and projective variety can be endowed with a certain action of $\mathbb{C}^\times$ playing the role of the *Hodge decomposition*. Finally, a Tannakian point of view on this construction will be given.

6.1 The schematization problem and one solution

In *pursuing stacks* Grothendieck considers what he calls the *schematization problem*. The questions he asked are not very well defined and not very precise (e.g. he was considering higher stacks without even having defined higher groupoids !), which makes the understanding of the problem hazardous. I personally understood the *schematization problem* in the following way (I refer to the appendix of [To2] for more details).

The schematization problem

1. For a ring k, there should exist a notion of *schematic homotopy types over* k. These are expected to be ∞-*stacks* (i.e. presheaves of ∞-groupoids, or equivalently presheaves of homotopy types) on the site of k-schemes satisfying certain algebraicity conditions. It is expected for example that the Eilenberg-MacLane stack $K(\mathbb{G}_a, n)$ is a schematic homotopy type. It is also expected that the category of schematic homotopy types is stable by certain standard constructions as fiber products, extensions ...

2. For any topological space X, and any ring k, there should exist a schematic homotopy type $(X \otimes k)^{sch}$, called the *schematization of X over k*. The stack $(X \otimes k)^{sch}$ is required to be the universal schematic homotopy type receiving a morphism from X (considered as a constant presheaf of homotopy types and therefore as a constant ∞-stack).

3. If $k = \mathbb{Q}$, simply connected schematic homotopy types over $\mathbb{Q}$ are expected to be models for simply connected rational homotopy types. Furthermore, for any simply connected space X, the schematization $(X \otimes \mathbb{Q})^{sch}$ is expected to be a model for the rational homotopy type of the space X as defined by D. Quillen and D. Sullivan.

In [To2] I propose two solutions to the schematization problem. I will present here the second one, which is for the moment only valid when the base ring k is a field.

In order to state the main definition, let us recall that for any ring k one has the site of affine k-schemes Aff/k endowed the ffqc (faithfully flat and quasi-compact) topology. The Segal category of stacks on the site Aff/k will be denoted by $LAff_k^{\sim,ffqc}$, and its object will be called *stacks over k*. Note that the homotopy category of $LAff_k^{\sim,ffqc}$ is simply the homotopy category of simplicial presheaves on Aff/k. Let us also recall that a stack over k, F, has a sheaf of connected component $\pi_0(F)$, and sheaves of homotopy groups $\pi_i(F,*)$ well defined for any global base point $* \longrightarrow F$.

Definition 6.1 *Let k be a field. A* neutral schematic homotopy type over k *is a stack $F \in LAff_k^{ffqc}$ which satisfies the following three conditions.*

1. *There exist a morphism of stacks $* \longrightarrow F$ inducing an isomorphism of sheaves $* \simeq \pi_0(F)$.*

2. *The homotopy sheaf $\pi_1(F,*)$ is representable by an affine group scheme (for any choice of global base point).*

3. *For any $i > 1$, the homotopy sheaf $\pi_i(F,*)$ is representable by an affine and unipotent group scheme.*

A schematic homotopy type over k *is a stack over k which after a field base change k'/k becomes a neutral schematic homotopy type over k'.*

A solution to the schematization problem of Grothendieck is given by the following theorem.

Theorem 6.2 *([To2, Thm. 3.3.4]) Let k be any field. Let X be any connected simplicial set, considered as a constant simplicial presheaf and therefore as an object in $Ho(k)$. There exist a schematic homotopy type $(X \otimes k)^{sch}$ over k (automatically neutral), and a morphism $X \longrightarrow (X \otimes k)^{sch}$ which is universal among morphisms towards schematic homotopy types over k.*

For a connected simplicial set X and $x \in X$ one can use the universal property of the schematization in order to prove the following fundamental properties (see [To2]).

1. The sheaf $\pi_1((X \otimes k)^{sch}, x)$ is naturally isomorphic to $\pi_1(X,x)^{alg}$, the pro-algebraic hull of the discrete group $\pi_1(X,x)$ (relative to the field k).

2. For any finite dimensional local system of k-vector spaces L on X, the morphism $X \longrightarrow (X \otimes k)^{sch}$ induces isomorphism in cohomology with local coefficients
$$H^*((X \otimes k)^{sch}, L) \simeq H^*(X, L)$$
(this makes sense because of property (1) above; L can also be considered as a local system on the stack $(X \otimes k)^{sch}$).

3. Let assume that X is a finite simply connected simplicial set. Then, one has natural isomorphisms
$$\pi_i((X \otimes k)^{sch}, x) \simeq \pi_i(X,x) \otimes \mathbb{G}_a \quad if \quad car\, k = 0,$$
$$\pi_i((X \otimes k)^{sch}, x) \simeq \pi_i(X,x) \otimes \hat{\mathbb{Z}}_p \quad if \quad car\, k = p > 0.$$

4. The functor $X \mapsto (X \otimes k)^{sch}$, when restricted to rational (resp. p-complete) simply connected simplicial sets of finite type, is fully faithful.

Remark 6.3 Schematic homotopy types as defined in Def. 6.1 are very close to the notion of *Tannakian Segal gerbes* introduced in §5.3, and actually the two notions are more or less equivalent. In the same way, the object $(X \otimes k)^{sch}$ of theorem 6.2 is conjecturally the dual Tannakian gerbe of some Tannakian Segal category of perfect complexes of X. These relations will be precised later in §6.3.

6.2 Schematization and Hodge theory

A complex algebraic variety X has an underlying topological space of complex point with the analytic topology X^{top}. It is well known that the topology of X^{top} is in general not easy to describe in purely algebraic terms, as for example if X is defined over a number field K the topology of X^{top} depends non trivially on the embedding of K into $\mathbb{C}$. However, the schematization $(X^{top} \otimes \mathbb{C})^{sch}$ can be described purely in algebraic terms and without referring to the analytic topology (this is in fact an incarnation of the Riemann-Hilbert correspondence). In fact, as shown in [Ka-Pa-To1], the stack $(X^{top} \otimes \mathbb{C})^{sch}$ has an explicit algebraic model which uses algebraic de Rham complexes of forms with coefficients in various flat bundles. This explicit description in terms of differential forms allows one to use Simpson's non-abelian Hodge correspondence in order to endow the stack $(X^{top} \otimes \mathbb{C})^{sch}$ with a *Hodge decomposition*, which reflects how the algebraic nature of the manifold X interacts with his homotopy type.

Theorem 6.4 *(Katzarkov-Pantev-Toën, [Ka-Pa-To1]) Let X be a smooth projective complex algebraic variety. There exists a natural action of the discrete group $\mathbb{C}^{\times}$, called the Hodge decomposition, on the stack $(X^{top} \otimes \mathbb{C})^{sch}$ satisfying the following conditions.*

1. *The induced action on $H^*((X^{top} \otimes \mathbb{C})^{sch}), \mathcal{O}) \simeq H^n(X^{top}, \mathbb{C})$ is compatible with the Hodge decomposition (i.e. $\mathbb{C}^{\times}$ acts by weight q on the factor $H^p(X, \Omega^q_X)$).*
2. *The induced action of $\mathbb{C}^{\times}$ on $\pi_1((X^{top} \otimes \mathbb{C})^{sch}), x) \simeq \pi_1(X, x)^{alg}$ is the one constructed by C. Simpson in [Si2].*
3. *Assume that X is simply connected. The induced action of $\mathbb{C}^{\times}$ on $\pi_i((X^{top} \otimes \mathbb{C})^{sch})) \simeq \pi_i(X) \otimes \mathbb{G}_a$ is compatible with the Hodge decomposition on the rational homotopy type defined in [Mo].*

The above theorem gives a way to unify the Hodge decomposition on the rational homotopy type of [Mo] with the Hodge decomposition on the fundamental group of [Si2]. In a way, everything was already contained in the non-abelian Hodge correspondence [Si2], and the new feature of theorem 6.4 is to give a homotopy theory interpretation of this correspondence, based on the notion of schematic homotopy types and the schematization functor. The notion of schematic homotopy types was apparently the missing part in order to relate the various works on non-abelian Hodge theory (e.g. [Si3, Si4, Ha, Ka-Pa-Si2]) to actual homotopy theory.

Theorem 6.4 possesses two important consequences. First of all, it is not difficult to produce examples of finite CW complexes X such that the stack $(X \otimes \mathbb{C})^{sch}$ can not be endowed with a $\mathbb{C}^{\times}$-action satisfying the conditions of theorem 6.4. In particular, this gives new examples of homotopy types which are not realizable by projective manifolds, and the interesting new feature is that obstructions to realizability lie in higher homotopical invariants (precisely the action of the fundamental group on the higher homotopy groups, see [Ka-Pa-To1] for details). Another important consequence is the degeneracy of the *Curtis spectral sequence*, starting from the homology of $(X^{top} \otimes \mathbb{C})^{sch})$ with coefficients in the universal reductive local system, and converging to its homotopy groups (this is a generalization of the degeneracy of the Bar spectral sequence). To be a bit more precise, for any neutral and pointed schematic homotopy type $* \to F$, one can consider its universal reductive covering F^0, corresponding to the unipotent radical of $\pi_1(F, *)$. The stack F^0 is clearly a pointed and connected affine stack in the sense of [To2] and can therefore by represented as BG_*, where G_* is a simplicial affine group scheme such that each G_n is a free unipotent group scheme. By considering the central lower series of G_* one constructs a tower of fibrations

$$\dots B(G_*/\mathcal{C}^n G_*) \longrightarrow B(G_*/\mathcal{C}^{n-1} G_*) \longrightarrow \cdots \qquad B(G_*/[G_*, G_*]) = B(G_*)_{ab},$$

whose limit is the stack F^0. Corresponding to this tower of fibrations is a spectral sequence in homotopy as explained in [Bou-Kan], which by definition is the Curtis

spectral sequence of F. The $E_1^{p,q}$-term of this spectral sequence is $\pi_p(B(\mathcal{C}^q G_*/\mathcal{C}^{q+1}G_*)$, and its abutment is $\pi_*(F^0)$. Furthermore, one can see that $\pi_p(B(\mathcal{C}^q G_*/\mathcal{C}^{q+1}G_*))$ only depends on the graded group $\pi_*((G_*)_{ab})$ which is the homology of F^0, or in other words the homology of F with coefficients in the universal reductive local system $\mathcal{L}^{red}$. This implies that the E_1-term of the Curtis spectral sequence for F is given by the free Lie algebra over $H_{*>0}(F, \mathcal{L}^{red})$. The fact that when $F = (X^{top} \otimes \mathbb{C})^{sch}$, for a smooth projective complex manifold X, the Curtis spectral sequence degenerates at E_2 simply follows from some *weight property* of the action of $\mathbb{C}^\times$ on it. The conclusion is the following result.

Corollary 6.5 *(Katzarkov-Pantev-Toën, [Ka-Pa-To1])*

1. *For any pointed schematic homotopy type F, there exists a Curtis spectral sequence $E_r^{p,q}$, whose E_1-term is the free Lie algebra over $H_{*>0}(F, \mathcal{L}^{red})$ and whose abutment is $\pi_*(F)$.*

2. *Let X by a (pointed) smooth projective complex manifold. The Curtis spectral sequence of $(X^{top} \otimes \mathbb{C})^{sch}$ degenerates at E_2.*

Remark 6.6 Directly related to this is the formality theorem stated in [Ka-Pa-To1] which generalizes the well known fact that the rational homotopy type of a smooth projective manifold is formal.

The conclusion is that the existence of the Hodge decomposition of theorem 6.4 has very strong consequences on the schematization $(X^{top} \otimes \mathbb{C})^{sch}$ and this leads to a striking fact. In the general situation, the schematization $(X \otimes \mathbb{C})^{sch}$ of a space X seems very hard (if not impossible) to compute, as its homotopy invariants can be very far from the original homotopy invariants of X (see e.g. [To2, §3.4]). However, the degeneracy of the Curtis spectral sequence and the formality theorem imply that the schematization of a smooth projective manifold is much more simple than for an arbitrary space, and that one can expect to compute certain homotopy invariants which seem out of reach in the general situation. Another result going in the same direction is the fact that fundamental groups of Artin's neighborhood are *algebraically good groups* , and therefore that Zariski locally a smooth projective manifold has a very simple schematization. Together with the Van-Kampen theorem this again implies that the schematization of a smooth projective manifold looks more simple than for a general space (see [Ka-Pa-To2] for more on this).

6.3 The Tannakian point of view

For a space X, the fundamental group of its schematization $\pi_1((X \otimes k)^{sch}, x)$ is naturally isomorphic with the pro-algebraic completion of the discrete group $\pi_1(X, x)$ over the base field k. Hence, the functor $X \mapsto (X \otimes k)^{sch}$ is an extension of the pro-algebraic completion functor.

It is very well known that the pro-algebraic group $\pi_1(X,x)^{alg}$ has a Tannakian interpretation, as the Tannaka dual of the category of local systems of finite dimensional k-vector spaces on X (see [De2, §10.24]). In the same way the whole stack $(X \otimes k)^{sch}$ does have a Tannakian interpretation in the Segal sense of §5, at least conjecturally. Actually, it turned out that this Tannakian point of view was originally the way the schematization functor was introduced for the first time (in [To3]). This point of view has been also used in order to define the Hodge decomposition, way before the explicit construction presented in [Ka-Pa-To1] has been considered.

Let X be a connected finite CW complex, k be any ring (commutative with unit), and $C(X,k)$ be the category of complexes of presheaves of k-modules on X. In the category $C(X,k)$ there is a notion of *local quasi-isomorphism* (morphisms inducing quasi-isomorphisms on each stalks), and $C(X,k)$ can be made into a model category for which equivalences are local quasi-isomorphisms. Furthermore, the tensor product of complexes makes $C(X,k)$ into a symmetric monoidal model category. By localization one gets a $\otimes$-Segal category $LC(X,k)$. By definition, the Segal category $LParf(X,k)$ is the full sub-Segal category of $LC(X,k)$ consisting of rigid objects (they are exactly the perfect complexes, i.e. are locally on X quasi-isomorphic to a constant complex of presheaves associated with a bounded complex of projective k-modules of finite type).

Pulling back from the point gives a morphism of $\otimes$-Segal categories

$$LParf(k) := LParf(*,k) \longrightarrow LParf(X,k)$$

making $LParf(X,k)$ into a rigid k-tensor Segal category. Furthermore, taking a base point $x \in X$, and considering the pull-back along $x : * \longrightarrow X$ gives rise to a k-tensor morphism

$$\omega_x : LParf(X,k) \longrightarrow LParf(k).$$

Clearly, $(LParf(X,k), \omega_x)$ satisfies the conditions of conjecture Conj. 5.13, and therefore should be a Tannakian Segal category.

When k is a field, one expects the following conjecture. Recall that one has full embedding

$$i : LAff_k^{\sim,ffqc} \longrightarrow LAff_{Sp^\Sigma}^{\sim,sffqc}/\mathbb{R}Spec\, Hk.$$

Conjecture 6.7 *Let k be a field and $(X \otimes k)^{sch} \in LAff_k^{\sim,ffqc}$ be the stack defined as in Thm. 6.2.*

1. *There is a natural morphism*

$$i(X \otimes k)^{sch} \longrightarrow FIB_k(LParf(X,k))$$

which is a P-equivalence (see Def. 5.11). In particular, the natural morphism of k-tensor Segal categories

$$LParf(X,k) \longrightarrow LParf(i(X \otimes k)^{sch})$$

is an equivalence.

2. *If k is of characteristic 0, then the natural morphism*

$$i(X \otimes k)^{sch} \longrightarrow FIB_k(LParf(X,k))$$

is an equivalence.

Conjecture 6.7 essentially states that the stack $(X \otimes k)^{sch}$ is the Tannakian dual of the Tannakian Segal category $LParf(X,k)$ [12]. This is the (conjectural) Tannakian interpretation of the schematization functor. It is important to note that the object $FIB_k(LParf(X,k))$ is itself defined for any commutative ring k, which gives (up to conjecture 6.7) an extension of the schematization functor over any base ring.

A first application of conjecture 6.7 and the whole Tannakian formalism, is an alternative construction of the Hodge decomposition of [Ka-Pa-To1], more in the style of the construction of the Hodge structure on the fundamental group given in [Si2]. Indeed, let X be a connected smooth and projective complex variety. One has two $\mathbb{C}$-tensor Segal categories $LParf(X^{top}, \mathbb{C})$, and $LParf(X_{Dol}, \mathcal{O})$. The first one is the Tannakian Segal category of perfect complexes of $\mathbb{C}$-vector spaces on the topological space of complex point X^{top}. The latter is defined to be the Segal category of complexes of quasi-coherent $\mathcal{O}$-modules on the stack X_{Dol} (i.e. complexes of quasi-coherent sheaves endowed with an integrable Higgs field, see [Si2, Si4, Si5]) which are cohomologically bounded and whose cohomology sheaves (which are Higgs coherent sheaves) are semi-stable Higgs bundles of degree 0 on X. The equivalence of tensor dg-categories given in [Si2, Lem. 2.2] can be enhanced as an equivalence of $\mathbb{C}$-tensor Segal categories

$$LParf(X^{top}, \mathbb{C}) \simeq LParf(X_{Dol}, \mathcal{O}).$$

Now, the discrete group $\mathbb{C}^\times$ acts on the stack X_{Dol}, and therefore on the Tannakian Segal category $LParf(X_{Dol}, \mathcal{O})$. Using the equivalence above one gets an action of $\mathbb{C}^\times$ on the Tannakian Segal category $LParf(X^{top}, \mathbb{C})$, and therefore on its Tannakian dual $(X^{top} \otimes \mathbb{C})^{top}$. This construction was the original idea of the construction of the Hodge decomposition on the schematization, and have been abandoned by using the more explicit approach taken in [Ka-Pa-To1]. I think however that it is interesting to keep the Tannakian point of view, it helps understanding things better, even if the Tannakian formalism for Segal categories is still conjectural.

To finish this comparison let me mention the following conjecture, establishing a relation between schematic homotopy types and Tannakian Segal gerbes.

[12] In positive characteristic the object $(X \otimes k)^{sch}$ is not exactly the dual of $LParf(X,k)$ but still determines it. In fact, as shown in [Ka-Pa-To2] $(X \otimes k)^{sch}$ is of the form $B\mathbb{R}Spec\, B$, for B a co-simplicial Hopf algebra. Viewing B as a H_∞-Hopf Hk-algebra, conjecture 6.7 predicts that $B\mathbb{R}Spec\, B \simeq FIB_k(LParf(X,k))$.

Conjecture 6.8 *Let k be a field and $F \in LAff_k^{\sim,ffqc}$ be a schematic homotopy type in the sense of Def 6.1.*

1. *The k-tensor Segal category $LPar f(i(F))$ is a k-Tannakian Segal category, and therefore $FIB_k(LPar f(i(F)))$ is a k-Tannakian gerbe.*

2. *There is a natural morphism*

$$i(F) \longrightarrow FIB_k(LPar f(i(F)))$$

which is a P-equivalence (see Def. 5.11). In particular, the natural morphism $i(F) \longrightarrow FIB_k(LPar f(i(F)))$ is universal among morphisms from $i(F)$ towards k-Tannakian Segal gerbes.

3. *If k is of characteristic 0, then the natural morphism*

$$i(F) \longrightarrow FIB_k(LPar f(i(F)))$$

is an equivalence. In particular $i(F)$ is a k-Tannakian Segal gerbe dual to the k-Tannakian Segal category $LPar f(i(F))$. Furthermore, the morphism $F \mapsto i(F)$ induces and equivalence between the Segal category of schematic homotopy types and the Segal category of k-Tannakian Segal gerbes.

Conjecture 6.8 essentially says that schematic homotopy types are duals to Tannakian Segal categories, at least in the characteristic zero case. In positive characteristic however, schematic homotopy types are only approximations of Tannakian Segal gerbes, and the morphism $F \mapsto i(F)$ does not provide an equivalence between the two notions anymore. This last morphism is actually very similar to the normalization functor $N : Ho(Alg_k^{\Delta}) \longrightarrow Ho(E_\infty - Alg_k)$, going from the homotopy category of co-simplicial commutative k-algebras to the homotopy category of E_∞-k-algebras, which is well known not to be an equivalence (not even fully faithful) when k is of positive characteristic (see for example [Kr]). As co-simplicial algebras are strict forms of E_∞-algebras, schematic homotopy types are strict form of Tannakian Segal gerbes.

6.4 Other homotopy types in algebraic geometry

Inspired by the case of complex projective varieties and their Hodge decomposition, one can also define and study various schematic homotopy types related to other cohomological theories, as the l-adic and crystalline theories. In practice these schematic homotopy types can be constructed using some formalism of equivariant co-simplicial algebras (see [To2, §3.5] for an overview), but they are conjecturally duals to natural Tannakian Segal categories in the same way that $(X \otimes k)^{sch}$ is conjecturally the dual to the Tannakian Segal category $LPar f(X, k)$. As the schematization $(X^{top} \otimes \mathbb{C})^{sch}$ is endowed with a Hodge decomposition, these new homotopy

types have additional structures as for example action of the Galois group of the ground field, or F-isocrystal structures, which are in general given by an action of some group. This action is expected to capture interesting geometric and arithmetic information, as for example a rational point of a variety will give rise to a natural homotopy fixed point on the corresponding homotopy type. Actually, the map sending a rational point to a homotopy fixed point can reasonably considered as a non-abelian version of the Abel Jacobi maps (see [To2, §3.5.3] for more details), and its seems to me a very interesting invariant to study in the future. But this is another story, and I will finish this mémoire here ....

A Some thoughts about n-category theory (a letter to P. May)

In this appendix, I reproduce a letter to Peter May, written during the fall 2001, for the purpose of some kind of NSF proposal around the general theme *higher category theory and applications*. We wrote it together with L. Katzarkov, T. Pantev and C. Simpson in order to explain our interests in higher categories and its applications to algebraic geometry. I thought that it is very much related to the main subject of this mémoire and that it could be of some use to present it in this appendix.

Dear Peter,

In this short note we would like to share some of our thoughts on higher categories. We will be mainly concerned with applications, and so we will adopt a utilitarian and pragmatic point of view. We will try to explain why having a unified theory of n-categories is of fundamental importance for the applications. Though not directly related to the *unification program*, we hope that these considerations will be of some help for your proposal.

It seems to us that the fundamental question in the subject is not about the correct definition of an n-category (always in the weak sense, and always with $n \in [0, \infty]$). In practice, at least for our interest in the subject, we have found it more important to ask the following question: What is the $(n+1)$-category of n-categories ? Of course, this sounds like a joke, because being able to define the $(n+1)$-category of n-categories requires a notion of $(n+1)$-categories, and therefore of n-categories (this is maybe one of the possible explanation of the word *Pursuing* in Grothendieck's *Pursuing stacks*). However, we hope the following few pages will clarify our point of view.

We begin by describing some of the examples of higher categories which are of interest to us. After that we explain why a model category approach to higher categories seems to be very well suited to treat these examples, and also for the study of higher categories themselves. Finally, we will discuss the relevance of a unified theory for the purpose of applications.

In the following, let us suppose we have a good theory of n-categories (again, n could be ∞). By this, we mean for example that there exist a 1-category of n-categories with strict functors, denoted by $n - Cat$, as well as a well defined $(n+1)$-category of n-categories with lax functors, denoted by $n - \underline{Cat}$. For two n-categories A and B, the n-category of lax functors between A and B will be denoted by $\underline{Hom}(A, B)$. By a *good theory* we also mean that all expected properties are satisfied. We apologize for not being very precise about which theory of higher categories we use, but in fact every construction we will consider can be made rigorous by employing the n-Segal categories of Tamsamani-Simpson (see [Ta, Hi-Si]). We also apologize in advance for not paying the proper attention to the set-theorical

complications that arise (most of the time at least two universes $\mathbb{U} \in \mathbb{V}$ have to be chosen).

Some examples of n-categories

Here are some examples of higher categories we are interested in. The first three are ∞-categories for which the i-morphisms are invertible (up to $(i+1)$-morphisms) as soon as $i > 1$ (in other words, the Hom's are ∞-groupoids). This type of higher categories is of fundamental importance to us.

Simplicial categories: Simplicial categories are categories enriched over the category of simplicial sets (i.e. their Hom sets are endowed with the additional structure of simplicial sets, such that compositions are morphisms of simplicial sets). These type of categories is very important as they are models for a certain kind of ∞-categories, and appear naturally in topology. More precisely, if T is a simplicial category, one can produce an ∞-category $\Pi_\infty(T)$ by keeping the same set of objects, and replacing the simplicial sets of morphisms in T by their ∞-fundamental groupoids[13]. In other words, for two objects x and y,

$$Hom_{\Pi_\infty(T)}(x,y) := \Pi_\infty(Hom_T(x,y)).$$

By this construction, every simplicial category will be considered as an object in $\infty - Cat$, or in $\infty - \underline{Cat}$. Some basic examples of this kind are:

- For a 1-category C, and $S \subset Mor(C)$ a sub-set of morphisms, one can consider the simplicial localization of Dwyer and Kan $L(C,S)$. The construction $(C,S) \mapsto L(C,S)$ is the left derived functor of the usual localization $(C,S) \mapsto S^{-1}C$. If one considers $L(C,S)$ as an ∞-category, then there exists a lax functor $C \longrightarrow L(C,S)$, which is universal for lax functors $f : C \longrightarrow A$, with A an ∞-category, and such that f sends S to equivalence in A.

  If M is a simplicial model category, and $Equiv$ is the sub-set of equivalence in M, then $LM := L(M, Equiv)$ is equivalent to the simplicial category of fibrant and cofibrant objects in M. When M is a non-simplicial model category, one can still compute the simplicial sets of morphisms in LM by using the mapping spaces of M.

- One defines $Top := LSSet$, the simplicial localization of the model category of simplicial sets. As an ∞-category, the objects of Top are the fibrant simplicial sets, the 1-morphisms are the morphisms of simplicial sets, the 2-morphisms are the homotopies, the 3-morphisms are the homotopies between homotopies ... and so on. The ∞-category Top seems to us as fundamental as the 1-category Set.

[13] We assume such a construction exists for our theory of higher categories

- For a ringed topos $(T, \mathcal{O}_T)$, one can consider $C(T)$, the category of complexes of sheaves of $\mathcal{O}_T$-modules in T. Consider further the simplicial localization $LC(T)$ of $C(T)$ with respect to the quasi-isomorphisms. As before, the objects of $LC(T)$ are the fibrant complexes of sheaves of $\mathcal{O}_T$-modules (for a reasonable model structure), the 1-morphisms are the morphisms of complexes, the 2-morphisms are the homotopies, the 3-morphisms homotopies between homotopies ... and so on.

Stacks of complexes: For a 1-category C, one can consider the $(n+1)$-category $\underline{Hom}(C^{op}, n-\underline{Cat})$. This is the $(n+1)$-category of n-prestacks on C. When C is a Grothendieck site, one can also consider the full sub-$(n+1)$-category $n-St(C)$ of $\underline{Hom}(C^{op}, n-\underline{Cat})$ consisting of *n-stacks*. These are the n-prestacks which satisfy the descent conditions for hyper-coverings (due to the lack of space we will not make this precise here).

For a scheme X, let $\mathcal{O}_X - Mod$ be the ringed topos of sheaves of $\mathcal{O}$-modules on the big site on X with the faithfully flat and quasi-compact topology. Taking the simplicial localization of each category of complexes $C(\mathcal{O}_X - Mod)$, one obtains an ∞-prestack on the category of schemes

$$\begin{array}{rccc} LC(-,\mathcal{O}): & Sch^{op} & \longrightarrow & n-\underline{Cat} \\ & X & \mapsto & LC(\mathcal{O}_X - Mod). \end{array}$$

The fact that this ∞-prestack is actually an ∞-stack was proved by A.Hirschowitrz and C.Simpson (see [Hi-Si]). This ∞-stack contains some very important full ∞-sub-stacks, as for example $\underline{Perf}$, the ∞-stack of perfect complexes of $\mathcal{O}$-modules. The ∞-stack of perfect complexes plays a crucial role in C.Simpson's non-abelian Hodge theory. For example, if X is a scheme over $\mathbb{C}$, one can associate to X its de Rham shape X_{DR} and Dolbeault shape X_{Dol}. The shapes X_{DR} and X_{Dol} are sheaves (and therefore ∞-stacks) on the big flat site of schemes. The following two ∞-categories

$$\underline{Hom}_{\infty-St(Sch)}(X_{DR}, \underline{Perf}), \qquad \underline{Hom}_{\infty-St(Sch)}(X_{Dol}, \underline{Perf}),$$

are of particular interest (here $\underline{Hom}_{\infty-St(Sch)}$ denotes the ∞-categories of morphisms in the ∞-category of ∞-stack). Actually, the main objects of study of the non-abelian Hodge theory are the ∞-stacks

$$\mathcal{HOM}_{\infty-St(Sch)}(X_{DR}, \underline{Perf}), \qquad \mathcal{HOM}_{\infty-St(Sch)}(X_{Dol}, \underline{Perf}),$$

(here $\mathcal{HOM}_{\infty-St(Sch)}$ denotes the ∞-stack of morphisms, i.e. the internal Hom in $\infty - St(Sch)$). The ∞-stacks of this type possess rich additional structures, like symmetric monoidal structures, linear structures, duality, rigidity ... In order to understand these structures properly one has to make sense of a very advanced

theory of ∞-categories. Note however, that all the ∞-categories appearing in these constructions come from simplicial categories.

Using these kind of constructions, C. Simpson has developed non-abelian Hodge theory, and already several results have been proven by him and his collaborators. Let us mention for example the construction of the non-abelian Hodge filtration, Gauss-Manin connection and the proof of its regularity (see [Si4, Si5]), the higher Kodaira-Spencer deformation classes (see [Si7]), the non-abelian (p,p)-classes theorem (see [Ka-Pa]), a non-abelian analogue of the density of the monodromy (see [Ka-Pa-Si1]), the notion of non-ablian mixed Hodge structure (see [Ka-Pa-Si2]), and some new restrictions on homotopy types of projective manifolds (see [Ka-Pa-To1]). All these results where guessed and proved using higher stack and/or higher category theory.

Homotopy Galois theory: Let us go back to the ∞-category Top, obtained as the simplicial localization of $SSet$ with respect to all equivalences. For a nice enough space X (e.g. a CW-complex), one can consider the ∞-category $Top(X)$ of locally constant stacks of ∞-groupoids on X. This is an ∞-version of the category of locally constant sheaves of sets on X and has many analogous properties. In particular, it was shown by B. Toën and G. Vezzosi (see [To1, To-Ve1]) that the ∞-category $Top(X)$ can be used to reconstruct the whole homotopy type of X[14] much in the same way as the fundamental groupoid $\Pi_1(X)$ of X can be reconstructed from the category of locally constant sheaves of sets on X. Moreover, there is an explicit formula:

$$\Pi_\infty(X) \simeq \underline{Hom}^{geom}(Top(X), Top),$$

where the right hand side is the full sub-∞-category of $\underline{Hom}(Top(X), Top)$ of *geometric* (i.e. which are exact and possess a right adjoint) lax functors. Combined with the observation that there exists a natural equivalence $Top(X) \simeq \underline{Hom}(\Pi_\infty(X), Top)$, the above formula can be rewritten as

$$\Pi_\infty(X) \simeq \underline{Hom}^{geom}(\underline{Hom}(\Pi_\infty(X), Top), Top).$$

In this last formula, it is interesting to note that the ∞-category Top plays the role of a *dualizing object*. This observation is the starting point of a research program on very general higher Tannaka dualities, for which the dualizing object is replaced by the ∞-stack of perfect complexes (see [To3]). Once again, one should note the all the ∞-categories involved in these considerations are associated to some simplicial categories, and are endowed with various additional structures (linear, monoidal, tensorial ...).

The reconstruction result quoted above is also the starting point of a generalization of the categorical Galois theory (as developed by A.Grothendieck) to the setting of ∞-categories. Such a theory may find applications in homotopy theory

[14] This statement was first mentioned by A.Grothendieck in one of his letters to L.Breen.

(e.g. a new point of view on pro-finite, pro-nilpotent ... localizations), as well as in algebraic geometry (e.g. in etale homotopy type of schemes).

Monoidal n-categories: We have already mentioned that there are examples of higher categories possessing interesting extra structures. The symmetric monoidal structures form an important type of such structures and deserve special attention. One can try to make sense of the theory of symmetric monoidal n-categories in the following way.

Let Γ be the category of pointed finite sets. Define the $(n+1)$-category of symmetric monoidal n-categories $n-Sym$ to be the full sub-$(n+1)$-category of $\underline{Hom}(\Gamma, n-\underline{Cat})$ consisting of lax functors $F : \Gamma \longrightarrow n-\underline{Cat}$ which satisfy the *Segal conditions* (i.e. the same conditions satisfied by Γ-spaces).

An important class of symmetric monoidal ∞-categories comprises the symmetric monoidal model categories. In particular, if M is such a model category, then the associated ∞-category LM possesses a natural symmetric monoidal structure. Using this, one immediately checks that our ∞-stacks $LC(-,\mathcal{O})$, $\underline{Perf}$ ...of complexes are in fact symmetric monoidal ∞-stacks. For example, this monoidal structure may be of some importance for the new higher stack interpretation of Grothendieck duality obtained recently by A.Hirschowitz.

The higher monoidal categories may be applicable to many other situations. For example they provide a new point of view on E_∞-ring structures. As this is a kind of strictification result, we will discuss this in the next section (see *Strictification and monoidal structures*).

Extended $TQFT$: For $n \in [0,\infty]$ consider the n-category $n-Cob$ of cobordisms in dimension less than n. We do not know of any rigourous construction of this category, but intuitively its objects are oriented 0-dimensional compact varieties, the 1-morphisms are the oriented 1-dimensional compact varieties with boundary, the 2-morphisms are oriented compact surfaces with corners and so on. The n-category $n-Cob$ has a symmetric monoidal structure induced by disjoint union of varieties.

Following J.Baez and J.Dolan, one can formulate the general purpose of extended topological quantum field theories as the study of (higher) categories of representations of $n-Cob$. Classically, one introduces the n-category $n-Vect$, of n-vector spaces. Heuristically it is defined by induction on n. First, one defines $1-Vect$ to be the 1-category of finite dimensional vector spaces (over some base field). This category possesses direct sums and a symmetric monoidal structure (the tensor product) which make it into a *rig*[15] 1-category. Inductively, if the rig $(n-1)$-category $(n-1)-Vect$ is defined, $n-Vect$ will be the n-category of $(n-1)$-categories which

[15] A rig n-category is an n-category equipped with two different symmetric monoidal structures ($\oplus$ and $\otimes$), such that the second one is distributive over the first one. The words 'rig' refers to 'ring' without 'n' (i.e. a ring without a substraction)

are free modules of finite type over the rig $(n-1)$-category $(n-1)-Vect$. In contrast with $n-Cob$, there is a rigorous construction (due to B.Toen) of $n-Vect$ which utilizes Tamsamani's n-categories. In particular, $n-Vect$ is constructed as a rig n-category, and hence as a symmetric monoidal n-category (if one forget the first monoidal structure).

By definition, the n-category of n-dimensional topological quantum field theories is then given by

$$n-TQFT := \underline{Hom}^{\otimes}(n-Cob, n-Vect).$$

Unlike previous examples, the higher categories appearing in $TQFT$ do not come from simplicial categories.

Model categories and n-categories of lax functors

The above examples of higher categories are based in an essential way on the existence of n-categories of lax functors, i.e the existence of the $(n+1)$-category $n-\underline{Cat}$. In this section we would like to present a general approach due to C.Simpson, which allows one to construct n-categories of lax functors (see [Hi-Si, Pe]). This approach uses model categories, which seems to be significant in at least two ways. Firstly, it is directly related to your proposal to use model categories to compare different theories of n-categories. Secondly, it gives a way to state and prove various strictification results, which are of fundamental importance for many applications. Actually, these strictification results may also be directly relevant to the comparison of different theories of n-categories.

Model categories of n-categories: The general idea is to define a model structure on the category $n-Cat$ for which the weak equivalences are precisely the equivalences of n-categories, and which is *internal* (i.e. $n-Cat$ is a symmetric monoidal model category for the monoidal structure given by direct products). In other words, we require the existence of internal Hom's together with a compatibility condition with the model structure. Unfortunately, it seems that such a model structure can not exist directly on $n-Cat$[16]. For example, the category $SimpCat$, of simplicial categories (which are models for ∞-categories for which the ∞-categories or morphisms are ∞-groupoids) possesses a model structure defined by B.Dwyer and D.Kan, which has the correct weak equivalence but which is not internal (because the product of two cofibrant objects is not cofibrant anymore). It seems that any approach to n-categories which is of operadic nature might have this problem.

In order to circumvent this difficulty, C.Simpson has introduced a notion of n-precategory. With his definition the Tamsamani n-categories (or more generally the n-Segal categories) can be viewed as n-precategories satisfying some special

[16] At least, the known or expected model structures producing theories of n-categories do not seem to be internal.

conditions. The relation between n-precategories and n-categories is very similar to the relation between prespectra and spectra, presheaves and sheaves, prestacks and stacks, pre-Γ-spaces and Γ-spaces, etc.. Remarkably, C.Simpson showed that the category $n - PrCat$, of n-precategories possesses an internal model structure.

From now on let $n - PrCat$ denote the model category of n-Segal precategories (here n must be finite), and let $\underline{Hom}_{n-PrCat}$ denote its internal Hom. We are not going to recall its definition, but let us mention that objects of $n - PrCat$ can be thought off as systems of generators and relations for n-Segal categories, which in turn are models for ∞-categories whose ∞-category of n-morphisms are ∞-groupoids (i.e. all i-morphisms are invertible up to $(i+1)$-morphisms for $i > n$).

An important property of $n-PrCat$ is that its fibrant objects are all n-Segal categories (the converse is not true), and that every object is cofibrant. Let $n - SeCat$ be the full sub-category of $n - PrCat$ consisting of n-Segal categories. Since the model category $n - PrCat$ is internal one can define the $(n+1)$-Segal category of n-Segal categories $n - \underline{SeCat}$, by taking fibrant objects in $n - PrCat$, and by forming the Hom's between them with values in $n - PrCat$. Therefore, $n - \underline{SeCat}$ is a category with values in $n - SeCat$, which can be seen in an obvious way as a $(n+1)$-Segal category. In other words, to compute the correct n-Segal categories of functors between two n-Segal (pre)categories A and B, one has to consider $\underline{Hom}_{n-PrCat}(A, B')$, where B' is a fibrant model for B. In conformity with tradition one writes $\mathbb{R}\underline{Hom}_{n-PrCat}(A, B)$ for $\underline{Hom}_{n-PrCat}(A, B')$.

This construction is actually more general. For any model category M, which is enriched over $n - PrCat$ (as a model category), one can define a $(n+1)$-Segal category $Int(M)$, whose objects are fibrant and cofibrant objects in M together with its Hom's with values in $n - SeCat$ coming from the enrichment.

Strictification: Let I be a $(n+1)$-precategory and let M be a cofibrantly generated model category enriched over $n - PrCat$. As the category $n - PrCat$ is acting on M, one can define the notion of a representation of I in M, or I-modules in M. These I-modules form a category M^I. Since M is cofibrantly generated, it is likely that M^I is again a cofibrantly generated model category for the levelwise model structure (i.e. fibrations and equivalence are defined levelwise). Furthermore, the model category M^I is naturally enriched over $n - PrCat$, and one can consider the associated $(n+1)$-Segal category $Int(M^I)$.

On the other hand, one can first consider $Int(M)$, and then the $(n+1)$-Segal category of functors $\mathbb{R}\underline{Hom}_{n-PrCat}(I, Int(M))$. The expected strictification theorem would be the existence of a natural equivalence of $(n+1)$-Segal categories

$$\mathbb{R}\underline{Hom}_{n-PrCat}(I, Int(M)) \simeq Int(M^I).$$

In applications, this theorem is very fundamental, and for example is of essential use for the few results mentioned in the first part of this note. Some essential special cases of this theorem are proven by A.Hirschowitz and C.Simpson (see [Hi-Si]), but

the general case is still a conjecture. It is important to stress that this particular case of the strictification theorem is used in the proofs of the few results mentioned above.

One of the really important consequences of the strictification theorem is the Yoneda lemma. Indeed, for any n-Segal category A C.Simpson has defined a Yoneda embedding

$$h : A \longrightarrow \mathbb{R}\underline{Hom}_{n-PrCat}(A^{op}, (n-1) - SeCat).$$

By definition the essential image of this embedding consists of *representable functors*. A way to define this essential image would be to show that every n-Segal category is equivalent to a strict category with values in $(n-1) - PrCat$. For such a category, there is an obvious morphism of n-precategories $A \times A^{op} \longrightarrow (n-1) - PrCat$, which induces the functor h by adjunction. However, the fact that h is fully faithful is still a conjecture for $n > 1$ (it is proved for 1-Segal categories, see [Si1])).

As $(n-1) - SeCat$ is by definition equivalent to $Int((n-1) - PrCat)$, one may actually find a fully faithful functor

$$h : A \longrightarrow Int((n-1) - PrCat^{A^{op}}),$$

which shows that every n-Segal category A should embeds into a n-Segal category of the form $Int(M)$, for M an $(n-1) - PrCat$-enriched model category.

A direct consequence of this fact is the following method for *computing* the n-Segal categories of lax functors. Let A and B be two n-Segal categories, and let us consider the model category $(n-1) - PrCat^{A \times B^{op}}$. Then, the n-Segal category $\mathbb{R}\underline{Hom}_{n-PrCat}(A, B)$ can now be identified with the full sub-n-Segal category of $Int((n-1) - PrCat^{A \times B^{op}})$, consisting of functors $F : A \times B^{op} \longrightarrow (n-1) - PrCat$ such that each evaluation at an object a in A, $F_a : B^{op} \longrightarrow (n-1) - PrCat$ is equivalent to a *representable functor*. This gives a way to systematically reduce the computation to the case of n-Segal categories of the form $Int(M)$, which is again a very powerful tool for applications.

Unfortunately, it seems unlikely that the general approach of C.Simpson could be generalized or imitated for other theories of n-categories. Indeed, we have already mentioned that the model category of simplicial categories is not internal, and as far as we know the expected model structures for other theories may have the same problem. But of course, we do not have a proof that these model structures can not exist, and this is just a general feeling.

Strictification and monoidal structures: To finish this section, let us mention another conjectural strictification result related to E_∞-ring structures.

For this, let M be a symmetric monoidal closed model category, which is assumed to satisfy some nice but reasonable properties (e.g. is cofibrantly generated and left proper). One can make sense of E_∞-algebras in M. These form a category denoted by $E_\infty - Alg(M)$. It was shown by M.Spitzweck that the category $E_\infty - Alg(M)$ is almost a model category (for its natural model structure), and actually is

a model category in many cases. Whatever it is, there is a natural notion of a weak equivalence in $E_\infty - Alg(M)$, and so one can consider the ∞-category $LE_\infty - Alg(M)$, obtained by applying the simplicial localization of Dwyer and Kan to the set of weak equivalences.

On the other hand, one can consider first LM, which is a symmetric monoidal ∞-category, and the ∞-category

$$Comm(LM) := \underline{Hom}^{\otimes}(FS, LM)$$

of commutative (unital and associative, as usual) monoids in it. Here FS is the symmetric monoidal category of finite sets (where the monoidal structure is given by disjoint union).

It was shown by B. Toën (see [To3]) that there exists a natural functor

$$Comm(LM) \longrightarrow LE_\infty - Alg(M),$$

which is conjectured to be an equivalence of ∞-categories. This conjecture would identified E_∞-algebra as commutative monoids in certain ∞-categories (this point of view was also considered by T.Leinster).

The need of a unification

What we wanted to stress out in the previous discussions is that for purposes of applications, a good theory of n-categories requires in a very essential way the existence of n-categories of lax functors, as well as some strictification results. This is the reason why we have been very much interested in n-Segal categories instead of any other theories, for which we know, or at least we expect, the required properties to hold. Of course, there is a price to pay for using n-Segal categories, and this is what we would like to discuss in this last part.

When we look at T.Leinster's list of definition of n-categories, the definitions seem to belong to two different classes. There are definitions of n-categories for which the composition of morphisms is well defined but associativity only holds up to coherent homotopies, and definitions for which composition is defined only up to an equivalence (i.e. composition is not defined in the conventional sense). Simplicial categories, and all operadic definitions seem to belong to the first class, and Tamsamani-Simpson definitions to the second one. For the purpose of application and concrete manipulation each of these classes has its own advantages.

- The big advantage of the first type of definitions is that composition being well defined allows a more easy manipulation of n-categories themselves, and is also closer to the usual intuition of categories. Also, they can be easily generalized to general contexts, for example to deal with linear higher categories, or more exotic enrichment. For example, in your talk at Morelia you are already dealing with a general base category. Such notions of linear, or enriched

higher categories are much more unclear for the second type of definitions, and it may even be that there are no reasonable analogs.

On the other side, already the example of simplicial categories seem to show that defining higher categories of lax functors is quite difficult in this context. It also shows that even if one can hope to define these categories of lax functors, they could be difficult to use in practice.

- We have already mentioned the advantage of the second type of definitions, which is the existence of a good theory of internal Hom, and already explained this is of fundamental importance, even to define the objects we would like to study. The main problem one encounter using these kind definitions is the lack of computability. For example, when one wants to compute the n-category of functors $\underline{Hom}(A, B)$, between two n-Segal categories, one need first to consider a fibrant replacement A' of A. Usually, this fibrant replacement is highly non-explicit, and therefore very *un-computable*. This fact makes the standard categorical techniques (e.g. the Yoneda embedding) difficult to use in this context.

We have already mentioned that various higher categories (or higher stacks) we have encountered in applications are endowed with additional structure. For example, fur the purpose of higher Tannakian duality and non-abelian Hodge theory the stack of perfect complexes $\underline{Perf}$ has to be considered as a stack of *tensor* ∞-categories. Therefore, the first class of definitions of higher categories could be useful for us to give a sense to these additional structures (as the linearity for example). On the other hand, internal Hom's and strictifications results can not be avoided to manipulate, define and even prove things. This makes the second class of definitions of higher categories difficult to avoid for the purpose of our applications. The same kind of remark can be made concerning the monoidal structure. We have already mentioned one approach to monoidal structures in this letter, which is very much suited when one use the second class of definitions. There exist also another approach, which consist of considering E_∞-algebras in some model categories of higher categories, and which is more in the style of the second class of definitions. Again, the two approaches have their own advantages and dis-advantages, and probably both are needed to really prove non-trivial theorems.

This situation with the stack $\underline{Perf}$ presented here is only one particular example, and one can find many other situations where the two classes of definitions of higher categories seem to be needed. Therefore, for us, the unification of these two classes of definitions would be a great progress as far applicability is concerned. It would allow one to choose the most suited of the two models for each particular situation. Already for the theories of simplicial categories and 1-Segal categories, which are know to be equivalent, this principle is highly used, and without such an equivalence many results presented in this letter would have been unreachable to us. There is no doubt that a unified theory will make accessible some expected results

in algebraic geometry concerned with higher categorical structures.

These are the reasons why we are very much interested in participating in a unification program of all the different theories of higher categories, and this is why we believe that such a unification may also be useful to many other end-users of higher categories.

All the very best,

L.Katzarkov, T.Pantev, C.Simpson, B.Toen.

B Comparing various homotopy theories

As stated in section 1, there exist many different theory in order to do homotopy theory. Motivated by P. May's project to unify the various point of views on higher category theory, I have been interested in comparing four of these theories which I was interested in. They are the theories of S-categories, Segal categories, complete Segal spaces and quasi-categories. They are supposed to be all equivalent to each others, and in this appendix I present comparison adjunctions which are conjecturally Quillen equivalences.

Notations

The category of Segal precategories, $PrCat$, is the category of functors

$$A : \Delta^{op} \longrightarrow SSet$$

such that A_0 is a discrete simplicial set. It is endowed with the model structure described in [Hi-Si, Pe].

The model category of quasi-categories, $QCat$, is the category of simplicial sets, endowed with the model structure defined by Joyal.

The model category of complete Segal spaces, CSS, is the category of functors

$$X : \Delta^{op} \longrightarrow SSet$$

endowed with the model structure defined in [Re1].

Finally, $S - Cat$ is the category of S-categories (i.e. simplicially enriched categories). It is endowed with the model structure described in [Dw-Hir-Ka, XII-48][17].

[17]The proof given there is not correct, as the generating trivial cofibrations are not even equivalences. However, it seems that the model structure still exists.

Segal categories and quasi-categories

One defines a functor

$$\phi : PrCat \longrightarrow QCat$$

by the following way. For $A \in PrCat$, the simplicial set $\phi(A)$ is

$$\begin{array}{rccc} \phi(A): & \Delta^{op} & \longrightarrow & Set \\ & [n] & \mapsto & (A_n)_0. \end{array}$$

Let h_n be the object in $PrCat$ which is represented by $[n] \in \Delta$. This defines a cosimplicial object $[n] \mapsto h_n$, in $PrCat$. One has

$$\phi(A) \simeq Hom(h_-, A).$$

This shows that the functor ϕ possesses a left adjoint

$$\psi : QCat \longrightarrow PrCat.$$

More precisely, for $X \in QCat$, $\psi(X)$ is the co-equalizer of the diagram in $PrCat$

$$\coprod_{[m]\to[p]\in\Delta} X_p \times h_m \rightrightarrows \coprod_{[n]\in\Delta} X_n \times h_n.$$

This defines an adjunction

$$\psi : QCat \longrightarrow PrCat \qquad QCat \longleftarrow PrCat : \phi.$$

Conjecture B.1 *The previous adjunction is a Quillen equivalence.*

Segal categories and complete Segal spaces

For any $[n] \in \Delta$ one denotes by $\overline{I}(n)$ the category with $n+1$ objects and a unique isomorphism between them. Functors $\overline{I}(n) \longrightarrow C$ are then is one-to-one correspondence with strings of n composable isomorphisms in C.

Considering categories as object in $PrCat$ (via their nerves), one gets a cosimplicial object

$$\begin{array}{ccc} \Delta^{op} & \longrightarrow & PrCat \\ n & \mapsto & \overline{I}(n). \end{array}$$

This functor extends in a standard way to a functor

$$\Pi : SSet \longrightarrow PrCat.$$

We define a simplicial structure on $PrCat$ via the functor Π. For A and B objects in $PrCat$, the simplicial set of morphisms $Hom_s(A, B)$ is given by the following formula

$$Hom_s(A, B)_n := Hom(A \times \overline{I}(n), B).$$

Tensors and co-tensors are define by

$$X \otimes A := \Pi(X) \times A \qquad A^X := Hom_s(\Pi(X), A),$$

for any $A \in PrCat$ and $X \in SSet$. The category $PrCat$ becomes a simplicial model category.

We define a functor

$$\phi : PrCat \longrightarrow CSS$$

sending A to the bi-simplicial set

$$\begin{array}{rccc} \phi(A): & \Delta^{op} & \longrightarrow & SSet \\ & [n] & \mapsto & Hom_s(h_n, A). \end{array}$$

Here, h_n is again the object of $PrCat$ represented by $[n] \in \Delta$.

The functor ϕ possesses a left adjoint ψ defined as follows. For $X \in CSS$, the object $\psi(X)$ is the co-equalizer of the diagram

$$\coprod_{[m]\to[p]\in\Delta} X_p \otimes h_m \rightrightarrows \coprod_{[n]\in\Delta} X_n \otimes h_n.$$

This defines an adjunction

$$\psi : CSS \longrightarrow PrCat \qquad CSS \longleftarrow PrCat : \phi.$$

Conjecture B.2 *The previous adjunction is a Quillen equivalence.*

Complete Segal spaces and S-categories

The category $S-Cat$ being a model category, one can find a cofibrant resolution functor Γ^* in the sense of [Ho, §5.2].

Let $I(n)$ be the category with $n+1$ objects and a unique morphism between them. Functors $I(n) \longrightarrow C$ are in one-to-one correspondence with strings of n composable morphisms in C.

We define a functor

$$\phi : S - Cat \longrightarrow CSS$$

as follows. For $T \in S - Cat$, the bi-simplicial set $\phi(T)$ is defined by

$$\begin{array}{rccc} \phi(T): & \Delta^{op} & \longrightarrow & SSet \\ & [n] & \mapsto & Hom(\Gamma^*(I(n)), T). \end{array}$$

This functor possesses a left adjoint $\psi : CSS \longrightarrow S - Cat$. For $X \in CSS$, the S-category $\psi(X)$ is the co-equalizer of the diagram

$$\coprod_{[m]\to[p]\in\Delta} X_p \otimes_\Gamma I(m) \rightrightarrows \coprod_{[n]\in\Delta} X_n \otimes_\Gamma I(n).$$

Here, $\otimes_\Gamma$ is the tensor product over Γ. It is defined for a simplicial set Y and an S-category T by the co-equalizer of the following diagram

$$\coprod_{[m]\to[p]\in\Delta} \coprod_{Y_p} \Gamma^m(T) \rightrightarrows \coprod_{[n]\in\Delta} \coprod_{Y_n} \Gamma^n(T).$$

This defines an adjunction

$$\psi : CSS \longrightarrow S-Cat \qquad CSS \longleftarrow S-Cat : \phi.$$

Conjecture B.3 *The previous adjunction is a Quillen equivalence.*

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