# Affine stacks, schematic homotopy types and Hodge theory

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January 23, 2002

In the first lecture I have presented the homotopy category of stacks $Ho(T)$, together with its enriched structures of simplicial derived $Hom$'s and homotopy limits and colimits. The purpose of this second lecture is to construct interesting examples of higher stacks in the setting of algebraic geometry, and give a concrete application to non-abelian Hodge theory.

The main references for this talk are [T] and [K-P-T]. Two other references related to the subject are [Gr] and [Si1].

From now, $T = (Aff/k)$ will be the site of affine schemes over a base ring $k$, and will be endowed with the faithfully flat and quasi-compact topology (i.e. a morphism of $k$-algebras $A \longrightarrow B$ corresponds to a covering in $(Aff/k)$ if it is faithfully flat). As the notion of fibers might be not so clear for the flat topology I refer to the original paper [Ja] for the precise definition of local equivalences in $SPr(T)$. As before, $Ho(T)$ will be the homotopy category of stacks on $T$, obtained from $SPr(T)$ by formally inverting local equivalences. All the materials presented in the first lecture generalize to the present case.

Before going on, let me mention that I will not take into account any kind of set theory and universes considerations.

## 1 Pursuing stacks

*Pursuing stacks* is a long and un-published manuscript of Grothendieck (see [Gr]) in which various relations between homotopy theory and higher stacks theory are investigated. One of the problem discussed in it is the *schematization problem* that I am going to describe[1].

*The schematization problem:* Given a nice topological space $X$ (e.g. a $CW$-complex) and a scheme $S$, construct a stack $X \otimes S$ over $S$ satisfying the following three properties.

1. The stack $X \otimes S$ must be *algebraic* in some sense.
2. The cohomology of the stack $X \otimes S$ with coefficients in $\mathbb{G}_a$ is isomorphic to the cohomology of the space $X$ with coefficients in the ring $\mathcal{O}(S)$. In formula
$$H^*(X, \mathcal{O}(S)) \simeq H^*(X \otimes S, \mathbb{G}_a).$$
3. When $S = Spec\,\mathbb{Q}$ the stack $X \otimes Spec\,\mathbb{Q}$ should be closely related to the rational homotopy type of the space $X$ (as defined for example in [Q]).

I am going to present two solutions for this problem. The first one is based on the notion of *affine stacks*, and seems closer to what was expected in [Gr] than the second solution based on *schematic homotopy types*, which is an extension of the original problem (by modifying condition (2)).

[1] The following description is based on my own understanding of the manuscript, and one might have different interpretations.

## 2 Affine stacks

As affine stacks are generalization of affine schemes we will start by reviewing one possible definition of affine schemes.

For a sheaf of sets $F \in Sh(T)$, one can consider its $k$-algebra of functions $\mathcal{O}(F) := Hom(F, \mathbb{G}_a)$. This defines a functor from the category $Sh(T)$ to the (opposite) category of commutative $k$-algebras

$$\mathcal{O} : Sh(T) \longrightarrow (k - Alg)^{op}.$$

This functor possesses a right adjoint

$$Spec : (k - Alg)^{op} \longrightarrow Sh(T),$$

which sends an algebra $A$ to the sheaf represented by the affine scheme $Spec\, A$. Now, the Yoneda lemma implies that $Spec$ is fully faithful, and by definition its image is the category of affine schemes. In other words, a sheaf $F \in Sh(T)$ is an affine scheme if and only if the adjunction morphism $F \longrightarrow Spec\, \mathcal{O}(F)$ is an isomorphism.

For a stack $F \in Ho(T)$, the algebra of functions on $F$ will be replaced by the algebra of cohomology of $F$ with coefficients in $\mathbb{G}_a$. To define it, one can always suppose (up to an isomorphism in $Ho(T)$) that the presheaf $F_n$ of $n$-simplices of $F$ is represented by a disjoint union (maybe infinite) of affine schemes. Then, applying the functor $\mathcal{O}$ to each $F_n$ one gets a co-simplicial $k$-algebra

$$\begin{array}{rccc} \mathcal{O}(F) : & \Delta & \longrightarrow & k - Alg \\ & [n] & \mapsto & \mathcal{O}(F_n). \end{array}$$

One proves that the functor $F \mapsto \mathcal{O}(F)$ sends local equivalences to quasi-isomorphisms of co-simplicial algebras[2]. The functor $\mathcal{O}$ can then be derived and gives rise to a functor

$$\mathbb{L}\mathcal{O} : Ho(T) \longrightarrow Ho(k - Alg^{\Delta})^{op},$$

where the category $Ho(k - Alg^{\Delta})$ is obtained from the category of co-simplicial $k$-algebras $k - Alg^{\Delta}$ by formally inverting quasi-isomorphisms.

**Proposition 2.1** *The functor $\mathbb{L}\mathcal{O}$ possesses a right adjoint*

$$\mathbb{R}Spec : Ho(k - Alg^{\Delta}) \longrightarrow Ho(T)$$

*that is fully faithful.*

**Definition 2.2** *An affine stack is a stack in the essential image of $\mathbb{R}Spec$ (i.e. isomorphic to some $\mathbb{R}Spec\, A$ for some $A \in Ho(k - Alg^{\Delta})0$.*

*Examples:*

- An affine scheme is an affine stack.
- For any integer $n$, the stack $K(\mathbb{G}_a, n)$ is affine.
- For any integer $n > 0$, the stack $K(\mathbb{G}_m, n)$ is *not* an affine stack.
- The sub-category of affine stacks in $Ho(T)$ is the smallest full sub-category containing the stacks $K(\mathbb{G}_a, n)$ for all $n \geq 0$, and which is stable by homotopy limits.

[2] A morphism of co-simplicial k-algebra is a quasi-isomorphism if the induced morphism on the normalized co-chain complex is a quasi-isomorphism (see [T]).

*Remarks:*

- In general, an affine stack is a non-truncated stack. This means that it can only be considered as an $\infty$-stack. Furthermore, the truncation of an affine stack is in general not affine.
- As affine stacks are generalization of affine schemes, one can try using them as atlases to define a new notion of algebraic stacks in the category $Ho(T)$. This would allows to extend the notion of *geometric n-stacks* of [Si2] to the case of un-truncated stacks (or $\infty$-stack). We will see that *schematic homotopy types* are of this kind.
- If $k$ is of characteristic zero, then the homotopy category of co-simplicial algebras $Ho(k - Alg^{\Delta})$ is equivalent to the homotopy category of commutative differential graded $k$-algebras $Ho(k-cdga)$[3]. The previous proposition implies in this case that the category $Ho(k-cdga)$ embeds as a full sub-category in $Ho(T)$.

When the base ring $k$ is a field, and if we restrict ourselves to connected and pointed stacks, then one has the following characterization of affine stacks.

**Theorem 2.3** *A pointed and connected stack $F$ over a field $k$ is an affine stack if and only if its homotopy sheaves*[4] *$\pi_i(F,*)$ are representable by affine and unipotent group schemes.*

As an important corollary of this theorem one gets an equivalence between the homotopy categories of augmented co-simplicial $k$-algebras (or commutative $k$-dga when $k$ is a field of characteristic zero) and the full sub-category of connected and pointed stacks whose homotopy sheaves are affine and unipotent group schemes.

Let us finish this first section by a first solution to the schematization problem.

**Theorem 2.4** *For any simplicial set $X$ and any ring $k$, let still denote by $X \in Ho(T)$ the stack defined by the constant simplicial presheaf with values $X$. Then, there exist a affine stack $(X \otimes k)^{uni}$, and a morphism $X \longrightarrow (X \otimes k)^{uni}$ which is universal for morphisms into affine stacks. In particular one has $H^*(X,k) \simeq H^*((X \otimes k)^{uni}, \mathbb{G}_a)$.*

*Remarks:*

- Such a affine stack $(X \otimes k)^{uni}$ is called an *affination of $X$ over $k$*. It seems very close to the *schematization* discussed in [Gr]. More precisely, our affine stacks seems close the *complex of unipotent bundles* of [Gr, p. 446].
- When $k = \mathbb{Q}$ or $k = \mathbb{F}_p$ one can reprove and reinterpret all the usual well known results concerning rational and $p$-adic theory using the construction $X \mapsto (X \otimes k)^{uni}$ (see [T, §2.5]). However, the theory of affine stacks over $Spec\,\mathbb{Q}$ and $Spec\,\mathbb{F}_a$ is different from the theory of rational and $p$-compete spaces.
- If two 1-connected and finite $CW$-complexes $X$ and $Y$ are such that $(X \otimes \mathbb{Z})^{uni} \simeq (Y \otimes \mathbb{Z})^{uni}$ then $X$ and $Y$ are weakly equivalent. Furthermore, the functor $X \mapsto (X \otimes \mathbb{Z})^{uni}$ embeds the homotopy category of 1-connected and finite $CW$-complex as a faithful (but not full) sub-category of $Ho(T)$.

[3] We are using here positively graded co-chain complexes.

[4] Let us recall that for a pointed stack $F$, the homotopy sheaves $\pi_i(F,*)$ are by definition the sheaves associated to the presheaves

$$\begin{array}{rccc} \pi_i^{pr}(F,*): & (Aff/k)^{op} & \longrightarrow & Gp \\ & X & \mapsto & \pi_i(F(X),*). \end{array}$$

# 3 Schematic homotopy types

The main problem one can encounter working with affine stacks is that fundamental groups are unipotent group scheme (see 2.3). In particular, for a topological space $X$, the stack $(X \otimes k)^{uni}$ does not see that there might exists irreducible local systems of higher rank on $X$ (as an irreducible local system which is unipotent is automatically trivial). The notion of *schematic homotopy types* is precisely introduced to correct this problem and to allow general affine group schemes as fundamental groups.

From now on $k$ will be any field.

**Definition 3.1** *A (pointed and connected) schematic homotopy type $F$ is a pointed and connected stack $* \longrightarrow F$ satisfying the following two conditions.*

1. *The loop stack $\Omega_* F := * \times^h_F * \in Ho(T)$ is an affine stack.*
2. *The fundamental group sheaf $\pi_1(F, *)$ is represented by an affine group scheme.*

*Remarks:*

- Conjecturally condition (1) of the previous definition implies condition (2).
- Schematic homotopy types are analogs of classifying stacks of affine group schemes. Indeed, if $F = K(G, 1)$, for $G$ an affine group scheme, then $\Omega_* K(G, 1) \simeq G$ is an affine scheme and therefore an stack. In general schematic homotopy types are classifying stack of *affine group stack*. This also shows that they are quotient of $Spec\, k$ by the trivial action of an affine group stack, and in particular that they are *geometric* (or *algebraic*) if one admits affine stacks as atlases.

The following result characterizes the schematic homotopy types using their homotopy sheaves.

**Theorem 3.2** *A pointed and connected stack $F$ is a schematic homotopy type if and only if the sheaf $\pi_1(F, *)$ is represented by an affine group scheme and the sheaves $\pi_i(F, *)$ are represented by affine and unipotent group schemes for $i > 1$.*

In particular, connected very presentable stacks of [Si1] are schematic homotopy types. Inversely, any schematic homotopy type is pro-very-presentable (i.e. a projective limit of very presentable stacks). Very presentable stacks are also the *finitely presented* schematic homotopy types.

The second proposition for a solution to the schematization problem of [Gr] is the following theorem.

**Theorem 3.3** *For any pointed and connected simplicial set $X$ let still denote $X \in Ho(T)$ the connected and pointed stack defined by the constant simplicial presheaf with values $X$. Then, there exists a schematic homotopy type $(X \otimes k)^{sch}$ and a pointed morphism $u : X \longrightarrow (X \otimes k)^{sch}$ which is universal for morphisms into schematic homotopy types.*

Such a schematic homotopy type $(X \otimes \mathbb{C})^{sch}$ is called a *schematization of $X$ over $k$*.

*Remarks:*

- The universal property of $u$ is equivalent to the following two conditions.
    1. The category of $k$-linear representations (of finite dimension) of the affine group scheme $\pi_1((X \otimes k)^{sch}, *)$ is equivalent (through the functor $u^*$) to the category of local systems of $k$-vector spaces (of finite dimension) on $X$ (i.e. the category of $k$-linear representations (of finite dimension) of the discrete group $\pi_1(X, *)$).

2. For any linear representation $V$ of $\pi_1((X \otimes k)^{sch}, *)$ the morphism $u$ induces an isomorphisms
$$u : H^*((X \otimes k)^{sch}, V) \longrightarrow H^*(X, u^*(V)).$$
Another way to state this universal property is that $X$ and $(X \otimes k)^{sch}$ have the same local systems as well as the same cohomology with coefficients in them. The conditions that $(X \otimes k)^{sch}$ and $X$ have the same local system is not considered in [Gr] and is a new feature here.

- By the universal property, there always exists a morphism $(X \otimes k)^{sch} \longrightarrow (X \otimes k)^{uni}$ (a kind of maximal unipotent quotient), that is an isomorphism when $X$ is 1-connected.
- In general $(X \otimes k)^{sch}$ might be quite difficult to describe, if not impossible. For example, there exists finitely presented groups $\Gamma$ such that $(B\Gamma \otimes k)^{sch}$ is not trivial but simply connected !

# 4 Application to Hodge theory

Let now $X$ be a smooth and projective (connected) variety over $\mathbb{C}$, and let us consider the schematization of its underlying topological space $(X^{top} \otimes \mathbb{C})^{sch} \in Ho(T)$ (now $k = \mathbb{C}$). From the universal property one gets the following topological invariants.

1.
$$H^*((X^{top} \otimes \mathbb{C})^{sch}, \mathbb{G}_a) \simeq H^*(X, \mathbb{C})$$

2. When $X$ is 1-connected, then
$$\pi_i((X^{top} \otimes \mathbb{C})^{sch}, *)(\mathbb{C}) \simeq \pi_i(X, *) \otimes \mathbb{C}.$$

3.
$$\pi_1((X^{top} \otimes \mathbb{C})^{sch}, *) \simeq \pi_1(X, *)^{alg},$$
where $\pi_1(X, *)^{alg}$ is the pro-algebraic completion of $\pi_1(X, *)$ over $\mathbb{C}$.

In all cases there exist Hodge decompositions on these invariants (see [Mo] and [Si3] for (2) and (3)) which, as explained to us by Tony in his lecture, can be interpreted by actions of the group $\mathbb{C}^*$. The main theorem of [K-P-T] states that all these actions come from a natural action of $\mathbb{C}^*$ on the stack $(X^{top} \otimes \mathbb{C})$ itself.

**Theorem 4.1** *(Katzarkov, Pantev, - ) There exists a natural action of the group $\mathbb{C}^*$ on the stack $(X^{top} \otimes \mathbb{C})$ that induces the three previous Hodge decompositions.*

The action of $\mathbb{C}^*$ on the stack $(X^{top} \otimes \mathbb{C})$ has to be understood as a *Hodge decomposition*. It is part of a richer structure of *mixed schematic Hodge structure* on $(X^{top} \otimes \mathbb{C})$, including a weight filtration, a real structure and some purity conditions. One can deduce easily from the existence of this structure natural obstructions on the homotopy types of smooth and projective varieties, and construct explicit counter-examples.

# References


[Gr] A. Grothendieck, *Pursuing stacks*, un-published manuscript (1983 – 1984).

[Ja] J. F. Jardine, *Simplicial presheaves*, J. Pure and Appl. Algebra **47** (1987), 35 – 87.

[K-P-T] L. Katzarkov, T. Pantev, B. Toen, *Schematic homotopy types and non-abelian Hodge theory I*, preprint *math.AG*/0107129.

[Mo] J. W. Morgan, *The algebraic topology of smooth algebraic varieties*, Inst. Hautes Etudes Sci. Publ. Math. No. **48** (1978), 137 – 204.

[Q] D. Quillen, *Rational homotopy theory*, Ann. of Math. **90** (1969), 205 – 295.

[Si1] C. Simpson, *Homotopy over the complex numbers and generalized de Rham cohomology*, in Moduli of vector bundles (Sanda 1994, Kyoto 1994) 229 – 263, Lecture Notes in Pure and Appl. Math. **179**, Dekker, Nwe-York 1996.

[Si2] C. Simpson, *Algebraic (geometric) n-stacks*, preprint *math.AG*/9609014.

[Si3] C. Simpson, *Higgs bundle and local systems*, Publ. Math. I.H.E.S. **75** (1992), 5 – 95.

[T] B. Toen, *Schématization des types d'homotopie*, preprint *math.AG*/0012219.